

# GLOBAL WELL-POSEDNESS OF THE NLS SYSTEM FOR INFINITELY MANY FERMIONS

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**ABSTRACT.** In this paper, we study the mean field quantum fluctuation dynamics for a system of infinitely many fermions with delta pair interactions in the vicinity of an equilibrium solution (the Fermi sea) at zero temperature, in dimensions  $d = 2, 3$ , and prove global well-posedness of the corresponding Cauchy problem. Our work extends some of the recent important results obtained by M. Lewin and J. Sabin in [33, 34], who addressed this problem for more regular pair interactions.

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## 1. INTRODUCTION

There are two fundamental types of elementary particles in nature, bosons and fermions. The dynamics of a system of  $N$  bosons in  $\mathbb{R}^d$  interacting pairwise is described by solutions

to the Schrödinger equation on  $L^2(\mathbb{R}^{Nd})$

$$i\partial_t \psi_N = H_N \psi_N \quad , \quad \psi_N(0) = \psi_{N,0} \quad (1.1)$$

where

$$H_N = \sum_{j=1}^N (-\Delta_{x_j}) + \sum_{1 \leq j < k \leq N} U(x_j - x_k)$$

is the  $N$ -particle Hamiltonian; the precise form of the pair interaction potential  $U$  depends on the model at hand. Bosons satisfy Bose-Einstein statistics, that is, the bosonic wave function  $\psi_N$  is completely symmetric under exchange of particle variables  $x_j \in \mathbb{R}^d$ , i.e.,

$$\psi_N(x_1, \dots, x_N) = \psi_N(x_{\sigma(1)}, \dots, x_{\sigma(N)}) ,$$

for any permutation  $\sigma \in S_N$  (the  $N$ -th symmetric group). The simplest example of a completely symmetric state is a factorized state

$$\psi(x_1, \dots, x_N) = \prod_{j=1}^N \phi(x_j) .$$

It describes a configuration where all bosons are in the same quantum mechanical state  $\phi$ . This situation prominently occurs in Bose-Einstein condensation, a topic in mathematical research that has been extremely actively investigated in recent years, [2, 37]. For a bosonic system with mean field scaling  $U(x) = \frac{1}{N}V(x)$ , it can be proven rigorously (for  $V$  sufficiently regular) that in the limit  $N \rightarrow \infty$ , the mean field dynamics is characterized by a factorized state where  $\phi$  satisfies a nonlinear Hartree equation,

$$i\partial_t \phi = -\Delta \phi + (V * |\phi|^2)\phi .$$

If  $V = V_N$  depends on  $N$  and tends, in a suitable manner, to a delta distribution  $\delta$  as  $N \rightarrow \infty$ , it can be proven that the mean field dynamics is determined by a nonlinear Schrödinger equation,

$$i\partial_t \phi = -\Delta \phi + |\phi|^2 \phi ,$$

see [13, 15, 16, 17, 19, 20, 24, 26, 29, 32, 31, 40] and the references therein, and [41] for a survey.

In this paper, we will study the mean field dynamics of a system of infinitely many fermions, and are particularly interested in the situation where the pair interaction between particles are described by a delta potential. Fermions satisfy Fermi-Dirac statistics; while their dynamics is determined by a Schrödinger equation of the same form as (1.1), the wave function  $\psi_N$  is completely antisymmetric under exchange of particle variables,

$$\psi_N(x_1, \dots, x_N) = (-1)^{\text{sign}(\sigma)} \psi_N(x_{\sigma(1)}, \dots, x_{\sigma(N)})$$

for any permutation  $\sigma \in S_N$ . The simplest example of such a state is a Slater determinant

$$\psi(x_1, \dots, x_N) = \det \left[ u_i(x_j) \right]_{i,j=1}^N .$$

As a consequence of the antisymmetry, no two fermions can be in same quantum state  $u_j$ , otherwise  $\psi_N$  vanishes; in fact,  $\{u_j\}$  can generally be chosen to be an orthonormal family in  $L^2(\mathbb{R}^d)$ . This is the Pauli exclusion principle.

The ground state for a system of  $N$  non-interacting fermions on the unit torus  $\mathbb{T}^d$  with kinetic energy operator  $\sum_j (-\Delta_{x_j})$  has the following simple form. In frequency space,

$$\left( \psi_N, \sum_j (-\Delta_{x_j}) \psi_N \right) = \sum_{p_i \in \mathbb{Z}^d, i=1, \dots, N} \left( \sum_{j=1}^N p_j^2 \right) |\widehat{\psi}_N(p_1, \dots, p_N)|^2.$$

Therefore, it is easy to see that the lowest energy configuration is given by  $\widehat{\psi}_N = \bigwedge_{i \in F_N} \delta_i$  where  $F_N \subset \mathbb{Z}^d$  is the subset of the  $N$  distinct lattice points closest to the origin, and  $\delta_i$  is the Kronecker delta at  $i \in \mathbb{Z}^d$ . Clearly, the set  $F_N$  is an approximate ball  $B_{R_N}(0) \cap \mathbb{Z}^d$  of radius  $R_N \sim N^{1/d}$ , and it is referred to as the *Fermi sea*.

In this paper, we study the mean field quantum fluctuation dynamics for a system of infinitely many fermions with delta pair interactions in the vicinity of the Fermi sea at zero temperature, in dimensions  $d = 2, 3$ , and prove global well-posedness of the Cauchy problem. Our work extends some recent important results obtained by M. Lewin and J. Sabin, [33, 34], who address this problem for more regular pair interactions.

To describe the problem in more detail, we first consider a finite system of  $N$  fermions with mean field interactions involving a pair potential  $w$ . This system is described by  $N$  coupled Hartree equations

$$\begin{cases} i\partial_t u_1 = (-\Delta + w * \rho) u_1 & , & u_1(t=0) = u_{1,0} \\ \dots & & \dots \\ i\partial_t u_N = (-\Delta + w * \rho) u_N & , & u_N(t=0) = u_{N,0} \end{cases} \quad (1.2)$$

where  $\rho$  is the total density of particles

$$\rho(t, x) = \sum_{j=1}^N |u_j(t, x)|^2 \quad (1.3)$$

and  $w$  is the interaction potential. To account for the Pauli principle, the family  $\{u_{j,0}\}_{j=1}^N$  is assumed to be orthonormal, and it can be shown that  $\{u_j\}_{j=1}^N$  remains orthonormal for  $t > 0$ , as long as the Cauchy problem is well-posed in an appropriate space of solutions.

Systems similar to (1.2), but with a finite expected particle number  $\int \rho dx$  as  $N \rightarrow \infty$ , have been analyzed extensively in the literature, see for instance [1, 8, 9, 10, 12, 43]. Those systems describe a dilute gas with normalized density  $\rho(t, x) = \frac{1}{N} \sum_{j=1}^N |u_j(t, x)|^2$  (or  $\rho(t, x) = \sum_{j=1}^\infty \lambda_j |u_j(t, x)|^2$  with  $\lambda_j > 0$  and  $\sum \lambda_j = 1$ ) such that in the limit as  $N \rightarrow \infty$ ,  $\gamma$  is trace class. They can be derived rigorously from a combined mean field and semi-classical limit from a quantum system of interacting fermions; see [4, 3, 18, 25, 5, 38] and the references therein. In this limit, the exchange term is of a lower order of magnitude in powers of  $\frac{1}{N}$  than the main interaction term, therefore it does not appear in this analysis.

Corresponding to (1.2), one can introduce the one particle density matrix

$$\gamma_N(t) = \sum_{j=1}^N |u_j(t)\rangle\langle u_j(t)|, \quad (1.4)$$

which is the rank- $N$  orthogonal projection onto the span of the orthonormal family  $\{u_j(t)\}_{j=1}^N$ . Then the system (1.2) can be written in the density matrix form

$$i\partial_t \gamma_N = [-\Delta, \gamma_N] + [w * \rho_{\gamma_N}, \gamma_N] \quad (1.5)$$

$$\gamma_N(t=0) = \sum_{j=1}^N |u_{j,0}\rangle\langle u_{j,0}|, \quad (1.6)$$

with density

$$\rho_{\gamma_N}(t, x) = \gamma_N(t, x, x). \quad (1.7)$$

Orthonormality of the family  $\{u_j\}_{j=1}^N$  implies that  $0 \leq \gamma \leq 1$ .

For the system (1.2) - (1.3), respectively (1.5) - (1.7), the expected particle number  $\int \rho_N dx$  diverges in the limit  $N \rightarrow \infty$ , hence the one-particle density matrix  $\gamma = \sum_{j=1}^{\infty} |u_j\rangle\langle u_j|$  fails to be of trace class (but has bounded operator norm). The analysis of the dynamics of the fermion gas in this case becomes much more difficult. Lewin and Sabin, in [33, 34], were the first authors to address the behavior of (1.2) in this situation. The main problem is to properly give meaning to, and to understand solutions to the evolution equation

$$i\partial_t \gamma = [-\Delta, \gamma] + [w * \rho_{\gamma}, \gamma] \quad (1.8)$$

$$\gamma(t=0) = \gamma_0, \quad (1.9)$$

with  $\rho_{\gamma}(t, x) = \gamma(t, x, x)$ , where  $\gamma_0$  is not of trace class. To be consistent with the fact that the system models the behavior of infinitely many fermions, the Pauli principle is incorporated by requiring that  $0 \leq \gamma(0) \leq 1$ , thus  $\gamma$  has a bounded operator norm (among other properties). We note that the exchange term is in this situation again of lower order, and is therefore omitted.

In [33, 34], Lewin and Sabin study the dynamics of trace class perturbations  $Q := \gamma - \gamma_f$  around a non-trace class reference state  $\gamma_f$ . The latter is chosen to correspond to the Fermi sea of the non-interacting system. For inverse temperature  $\beta > 0$  and chemical potential  $\mu > 0$ ,  $\gamma_f$  is given by the Fermi-Dirac distribution

$$\gamma_f(x, y) = \int_{\mathbb{R}^d} \frac{e^{ip(x-y)}}{e^{\beta(p^2 - \mu)} + 1} dp = \left( \frac{1}{e^{\beta(-\Delta - \mu)} + 1} \right)(x, y). \quad (1.10)$$

while in the zero temperature limit,

$$\gamma_f = \Pi_{\mu}^{-} = \mathbf{1}_{(-\Delta \leq \mu)}. \quad (1.11)$$

Lewin and Sabin prove that the Cauchy problem for  $Q$  is globally well-posed in a suitable subspace of the space of trace class operators, provided that the pair interaction  $w$  is sufficiently regular.

In the work at hand, we extend the results of [33, 34] to the most singular case  $w = \delta$ , so that the potential term becomes  $\delta * \rho = \rho$ . The finite  $N$  analogue to (1.2) is the system of coupled NLS equations,

$$\begin{cases} i\partial_t u_1 = (-\Delta + \rho)u_1 & , & u_1(t=0) = u_{1,0} \\ \dots & & \dots \\ i\partial_t u_N = (-\Delta + \rho)u_N & , & u_N(t=0) = u_{N,0} \end{cases} \quad (1.12)$$

where

$$\rho(t, x) = \sum_{j=1}^N |u_j(t, x)|^2. \quad (1.13)$$

The family  $\{u_j\}_{j=1}^N$  remains orthonormal for  $t > 0$ , as long as (1.12) is well-posed.

To study the system in the limit  $N \rightarrow \infty$ , we employ the density matrix formalism as in [33, 34], and consider (1.8) with  $w \rightarrow \delta$ ,

$$i\partial_t \gamma = [-\Delta, \gamma] + [\rho, \gamma] \quad (1.14)$$

$$\gamma(t=0) = \gamma_0. \quad (1.15)$$

Again, the Pauli principle requires that  $0 \leq \gamma_0 \leq 1$ . For simplicity, we consider a reference state  $\gamma_f$  that corresponds to the Fermi sea of the non-interacting system at zero temperature, and chemical potential  $\mu > 0$ ,

$$\gamma_f = \Pi_\mu^- = \mathbf{1}_{(-\Delta \leq \mu)}. \quad (1.16)$$

We study perturbations

$$Q = \gamma - \Pi_\mu^-, \quad (1.17)$$

in an appropriate space of solutions, and establish global well-posedness for the Cauchy problem

$$i\partial_t Q = [-\Delta + \rho_Q, \Pi_\mu^- + Q] \quad , \quad Q(0) = Q_0 \quad (1.18)$$

in two and three dimensions. As a crucial new ingredient that allow us to extend the work of Lewin-Sabin [33] to the much more singular case of the delta function potential, we establish new Strichartz estimates for density functions and density matrices in Section 3. We remark that Lewin and Sabin used Strichartz-type estimates that they established in [33] in the case of positive temperature, but not in the zero temperature situation.

## 2. STATEMENT OF THE MAIN RESULT

In this section, we first introduce some notation and relevant operator spaces. Then, we present the main results and describe the strategy of the proof.

**2.1. Notation.** For simplicity of exposition, we assume that the chemical potential has the value  $\mu = 1$ .

We will denote by  $\mathfrak{S}^p$  the Schatten spaces

$$\|Q\|_{\mathfrak{S}^p} := \left( \text{Tr}|Q|^p \right)^{1/p} \quad (2.1)$$

for  $p \geq 1$ . We define the Banach space  $\mathcal{X}$  by the collection of self-adjoint operators on  $L^2$ , equipped with the norm

$$\|Q\|_{\mathcal{X}} = \|Q\|_{\text{Op}} + \sum_{\pm} \left\| |\Delta + 1|^{\frac{1}{2}} Q^{\pm\pm} |\Delta + 1|^{\frac{1}{2}} \right\|_{\mathfrak{S}^1}, \quad (2.2)$$

where  $Q^{\pm\pm} = \Pi_1^{\pm} Q \Pi_1^{\pm}$ ,  $\Pi_1^- = \mathbf{1}_{(-\Delta \leq 1)}$  and  $\Pi_1^+ = \mathbf{1}_{(-\Delta > 1)}$ .

For an operator  $Q \in \mathcal{X}$ , we denote

$$\text{Tr}_0(-\Delta - 1)Q := \sum_{\pm} \text{Tr} |\Delta + 1|^{\frac{1}{2}} Q^{\pm\pm} |\Delta + 1|^{\frac{1}{2}}, \quad (2.3)$$

and we call it the *relative kinetic energy* from the reference state  $\Pi_1^-$ . We note that for a finite-rank smooth operator  $Q$ ,

$$\text{Tr}_0(-\Delta - 1)Q = \text{Tr}(-\Delta - 1)Q.$$

For an operator  $Q \in \mathcal{X}$ , the relative kinetic energy can be expressed as the limit of  $\text{Tr}(-\Delta - 1)Q_n$  as  $n \rightarrow \infty$ , where  $\{Q_n\}_{n=1}^{\infty}$  is a sequence of finite-rank smooth operators such that  $Q_n \rightarrow Q$  in  $\mathcal{X}$  as  $n \rightarrow \infty$  (see Lemma 3.2 in [21]).

**The space of initial data.** We define the *relative energy space*, which will contain the initial data for our main global well-posedness result, as the collection of perturbations having finite operator norm and relative kinetic energy,

$$\mathcal{K} := \left\{ Q = \gamma - \Pi^- \in \mathcal{X} : 0 \leq \gamma \leq 1 \right\}. \quad (2.4)$$

For an operator  $Q \in \mathcal{K}$ , we define the *relative energy* of the NLS system by

$$\mathcal{E}(Q) := \text{Tr}_0(-\Delta - 1)Q + \frac{1}{2} \int_{\mathbb{R}^d} (\rho_Q)^2 dx. \quad (2.5)$$

*Remark 2.1.* We note that if  $Q \in \mathcal{K}$ , the relative kinetic energy is positive definite, and is well-defined (see Lemma A.1).

*Remark 2.2.* In this article, we restrict ourselves to two and three-dimensions, since otherwise, the relative energy is not well-defined in the relative energy space  $\mathcal{K}$ , because the potential energy is not bounded by the relative kinetic energy in other dimensions. Indeed, in the proof of Lemma A.1, the use of the Lieb-Thirring inequality (5.5) fails when  $d = 1$ , and the use of the generalized Sobolev inequality (5.6) fails when  $d \geq 4$ .

**The solution space.** The solution to the NLS system (1.18) that we obtain in this paper belongs to the space  $\mathfrak{Y}^1$  which is defined as follows. Given  $I \subset \mathbb{R}$ , we define the Banach space  $\mathfrak{Y}^\alpha(I)$  of solutions by

$$\|Q\|_{\mathfrak{Y}^\alpha(I)} := \|Q\|_{C_t(I; \text{Op})} + \|\Pi_2^+ Q\|_{\mathcal{S}^\alpha(I)} + \|\rho Q\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)}, \quad (2.6)$$

where  $\mathcal{S}^\alpha(I)$  is a Strichartz space of operator kernels with  $\alpha$  derivatives; its precise definition is given in (3.9), below. Moreover,  $\eta = \eta(d, \alpha) \geq 0$  is either small or zero, depending on  $d$  and  $\alpha$ , and is defined in (5.3).

**2.2. The main result and a description of the proof.** We now state our main theorem.

**Theorem 2.3** (Global well-posedness). *Let  $d = 2, 3$ . For initial data  $Q_0 \in \mathcal{K}$ , there exist arbitrarily large  $T^-, T^+ > 0$  and a unique global solution  $Q \in \mathfrak{Y}^1((-T^-, T^+))$  to the system (1.18). Moreover,  $Q(t) \in \mathcal{K}$  and the relative energy is conserved, i.e.,*

$$\mathcal{E}(Q(t)) = \mathcal{E}(Q_0) \text{ for all } t \in \mathbb{R}.$$

Our approach to proving Theorem 2.3 is motivated by the strategy that Lewin and Sabin used in [33]. However, in order to implement it in the case of the singular  $\delta$  potential considered in our paper, we introduce new Strichartz estimates for density functions, and consequently need to employ different intermediate solutions spaces for our analysis. Similar arguments are used for the proof of energy conservation for solutions to dispersive nonlinear PDE in the energy space, see e.g. [11].

We will now summarize the key steps in our construction.

**Step 1:** In Section 5 we prove local well-posedness of the NLS system (1.18) in a space containing  $\mathcal{K}$ .

More precisely, we define the Banach space of initial data,  $\mathfrak{X}^\alpha$ , for  $\alpha \geq 1$ , by

$$\|Q\|_{\mathfrak{X}^\alpha} := \|Q\|_{\text{Op}} + \|\Pi_2^+ Q\|_{\mathcal{H}^\alpha}, \quad (2.7)$$

where  $\mathcal{H}^\alpha$  is the Hilbert-Schmidt type Sobolev space defined as follows:

$$\|Q\|_{\mathcal{H}^\alpha} := \|\langle \nabla \rangle^\alpha Q \langle \nabla \rangle^\alpha\|_{\mathfrak{S}^2} = \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha Q(x, x')\|_{L_x^2 L_{x'}^2}. \quad (2.8)$$

We note that

$$\mathcal{K} \subset \mathfrak{X}^\alpha \quad (2.9)$$

for any  $\alpha \geq 1$ .

Then our local well-posedness results can be stated as follows:

**Theorem 2.4** (Local well-posedness). *Let  $d = 2, 3$  and  $\alpha \geq 1$ . Given initial data  $Q_0 \in \mathfrak{X}^\alpha \cap \mathcal{K}$ , there exists an interval  $I$  and a unique solution  $Q \in \mathfrak{Y}^\alpha(I)$  to the equation (1.18).*

Crucial ingredients in the proof of this local well-posedness result are new Strichartz estimates for the density function  $\rho$  which we prove in Section 3 and in Proposition 5.1.

**Step 2:** In Section 6, we prove that the solutions constructed in Theorem 2.4 preserve the relative energy. More precisely, we prove the following:

**Theorem 2.5** (Conservation of the relative energy). *Let  $d = 2, 3$ . Let  $Q(t) \in \mathfrak{Y}^1(I)$  be the solution to the equation (1.18) with initial data  $Q_0 \in \mathcal{K}$  (constructed in Theorem 2.4). Then,*

$$\mathcal{E}(Q(t)) = \mathcal{E}(Q_0), \quad \forall t \in I. \quad (2.10)$$

The proof of Theorem 2.5 can be summarized as follows:

- (1) To begin with, we approximate the initial data  $Q_0 \in \mathcal{K}$  by a sequence  $Q^{(n)} \in \mathfrak{H}^2$  of very regular kernels, where

$$\|Q^{(n)}\|_{\mathfrak{H}^2} := \|\langle \nabla \rangle^2 Q^{(n)} \langle \nabla \rangle^2\|_{\mathfrak{S}^1}. \quad (2.11)$$

- (2) The local well-posedness for such regular solutions is established in Section 4. For the precise statement see Theorem 4.2, which gives the existence of a unique solution  $Q \in C_t(I; \mathfrak{H}^2)$  to the equation (1.18) for initial data  $Q_0 \in \mathfrak{H}^2$ .
- (3) Owing to their  $\mathfrak{H}^2$  regularity, energy conservation for such solutions can be proved in a straightforward manner, see Section 4. Hence, one obtains that every  $Q(t) \in C_t(I; \mathfrak{H}^2)$  which is a solution to the equation (1.18) for initial data  $Q_0$ ,

$$\mathcal{E}(Q(t)) = \mathcal{E}(Q_0), \quad \forall t \in I \quad (2.12)$$

is satisfied. This is formulated precisely in Proposition 4.3.

- (4) The last step in the proof of Theorem 2.5, which is carried out in Section 6, consists of showing that

$$\mathcal{E}(Q(t)) \leq \liminf_{n \rightarrow \infty} \mathcal{E}(Q^{(n)}(t)) = \liminf_{n \rightarrow \infty} \mathcal{E}(Q^{(n)}(0)) = \mathcal{E}(Q_0). \quad (2.13)$$

Repeating the same argument backwards in time, we obtain  $\mathcal{E}(Q_0) \leq \mathcal{E}(Q(t))$ . Thus, we conclude that  $\mathcal{E}(Q_0) = \mathcal{E}(Q(t))$ .

**Step 3:** Finally, in Section 7 we conclude the proof of Theorem 2.3 by showing that the local solution constructed in Theorem 2.4, which satisfies conservation of relative energy, can be extended for all times.



## 3. STRICHARTZ ESTIMATES

In this section, we present the main tools to prove local well-posedness of the NLS system (1.18) are new Strichartz estimates for operator kernels and for density functions. The latter are obtained in this paper using space-time Fourier transform techniques, which were instrumental in obtaining bilinear Strichartz estimates in the context of dispersive [6, 7] and wave [30] equations, and more recently, in the context of the Gross-Pitaevskii hierarchy [31, 15, 42] (which is an infinite hierarchy of coupled linear PDEs that describes the dynamics of infinitely many bosons; it appears in the derivation of the nonlinear Schrödinger equation from quantum many body systems).

We note that recently, in [22], Frank, Lewin, Lieb and Seiringer established Strichartz estimates for density functions of the form

$$\|\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}\|_{L_t^p(\mathbb{R}; L^q)} \lesssim \|\gamma_0\|_{\mathfrak{S}^{\frac{2q}{q+1}}}, \quad (3.1)$$

where  $p, q, d \geq 1$ ,  $1 \leq q \leq \frac{d+2}{d}$  and  $\frac{2}{p} + \frac{d}{q} = d$ . In particular, when  $\gamma_0 = \sum_{j=1}^N |\phi_j\rangle\langle\phi_j|$  is a projection operator with an orthonormal set  $\{\phi_j\}_{j=1}^N$  in  $L^2$ , the inequality (3.1) can be read as

$$\left\| \sum_{j=1}^N |e^{it\Delta}\phi_j|^2 \right\|_{L_t^p(\mathbb{R}; L^q)} \lesssim N^{\frac{q+1}{2q}}. \quad (3.2)$$

An important feature of (3.2) is that it improves summability compared to the trivial consequence of Strichartz estimates for the Schrödinger flow  $e^{it\Delta}$ ,

$$\left\| \sum_{j=1}^N |e^{it\Delta}\phi_j|^2 \right\|_{L_t^p(\mathbb{R}; L^q)} \leq \sum_{j=1}^N \|e^{it\Delta}\phi_j\|_{L_t^{2p}(\mathbb{R}; L^{2q})}^2 \lesssim \sum_{j=1}^N \|\phi_j\|_{L_x^2}^2 = N. \quad (3.3)$$

Indeed, the exponent  $\frac{q+1}{2q}$  on the right hand side of (3.2) is strictly less than one unless  $q = 1$ . In this sense, the Strichartz inequality (3.1) is a generalization of the kinetic energy inequality

$$\int_{\mathbb{R}^d} \left\{ \sum_{j=1}^N |\phi_j(x)|^2 \right\}^{1+\frac{2}{d}} dx \lesssim \int_{\mathbb{R}^d} \sum_{j=1}^N |\nabla \phi_j(x)|^2 dx, \quad (3.4)$$

which is dual to the famous Lieb-Thirring inequality [35, 36]. Later, in [23], Frank and Sabin extended (3.1) to the optimal range of  $q$ , that is,  $1 \leq q < \frac{d+1}{d-1}$ . In [33, 34], Lewin and Sabin employed these Strichartz estimates in their study on the Hartree equation for infinitely many fermions.

In this section, we introduce a different kind of Strichartz estimates for density functions of the form

$$\|\nabla|^{\frac{1}{2}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}\|_{L_t^2(\mathbb{R}; H^{\alpha_1})} \lesssim \|\langle \nabla \rangle^{\alpha} \gamma_0 \langle \nabla \rangle^{\alpha}\|_{\mathfrak{S}^2} \quad (3.5)$$

(see Theorem 3.3). We remark that compared to the Strichartz estimates (3.1), there is more gain in summability, equivalently a larger Schatten exponent ( $2 > \frac{2q}{q+1}$ ) on the right hand side, assuming more regularity on operators. Moreover, there is an improvement in regularity on the density function  $\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}$ . Indeed,  $\alpha_1 + \frac{1}{2} > \alpha$  in Theorem 3.3. This

fact will play a crucial role in proving our main theorem (see Lemma 5.4). The proof of Strichartz estimates (3.5) is based on the space-time Fourier transform, and is inspired by the proof of bilinear Strichartz estimates in Klainerman and Machedon [30, 31], Bourgain [6, 7], Chen and Pavlović [15] and Xie [42].

Another new ingredient in this article is Strichartz estimates for operator kernels (see Theorem 3.1), which enjoy the smoothing property of the Schrödinger flow  $e^{it\Delta}\gamma_0 e^{-it\Delta}$  from the Hilbert-Schmidt operators. Although they are quite natural for dispersive equations in the Heisenberg picture, to the best of the authors' knowledge, it is the first time that this kind of Strichartz estimates appear in the literature.

**3.1. Strichartz estimates for operator kernels.** For  $\alpha \geq 0$ , we define the *Hilbert-Schmidt Sobolev space*  $\mathcal{H}^\alpha$  by the collection of Hilbert-Schmidt operators, which are not necessarily self-adjoint, with the norm

$$\|\gamma_0\|_{\mathcal{H}^\alpha} := \|\langle \nabla \rangle^\alpha \gamma_0 \langle \nabla \rangle^\alpha\|_{\mathfrak{S}^2} = \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \gamma_0(x, x')\|_{L_x^2 L_{x'}^2}, \quad (3.6)$$

where  $\gamma_0(x, x')$  is the integral kernel of  $\gamma_0$ , i.e.,

$$(\gamma_0 g)(x) = \int_{\mathbb{R}^d} \gamma_0(x, x') g(x') dx'. \quad (3.7)$$

We say that an exponent pair  $(q, r)$  is *admissible* if  $2 \leq q, r \leq \infty$ ,  $(q, r, d) \neq (2, \infty, 2)$  and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \quad (3.8)$$

For a time-dependent operator  $\gamma(t)$  on an interval  $I \subset \mathbb{R}$ , we define its Strichartz norm by

$$\begin{aligned} \|\gamma(t)\|_{\mathcal{S}^\alpha(I)} := \sup_{(q,r): \text{admissible}} & \left\{ \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \gamma(t, x, x')\|_{L_t^q(I; L_x^r L_{x'}^2)} \right. \\ & \left. + \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \gamma(t, x, x')\|_{L_t^q(I; L_x^r L_{x'}^2)} \right\}. \end{aligned} \quad (3.9)$$

It is obvious that  $\mathcal{S}^\alpha(I) \hookrightarrow L_t^\infty(I; \mathcal{H}^\alpha)$ .

We identify the operator  $e^{it\Delta}\gamma_0 e^{-it\Delta}$  with its integral kernel

$$(e^{it\Delta}\gamma_0 e^{-it\Delta})(x, x') = (e^{it(\Delta_x - \Delta_{x'})}\gamma_0)(x, x'). \quad (3.10)$$

Then, as a function, the dispersive estimate for the linear propagator, one of whose spatial variables is frozen, yields the following Strichartz estimates.

**Theorem 3.1** (Strichartz estimates for operator kernels). *Let  $I \subset \mathbb{R}$ . Then, we have*

$$\begin{aligned} \|e^{it\Delta}\gamma_0 e^{-it\Delta}\|_{\mathcal{S}^\alpha(\mathbb{R})} & \lesssim \|\gamma_0\|_{\mathcal{H}^\alpha}, \\ \left\| \int_0^t e^{i(t-s)\Delta} R(s) e^{-i(t-s)\Delta} ds \right\|_{\mathcal{S}^\alpha(\mathbb{R})} & \lesssim \|R(t)\|_{L_t^1(\mathbb{R}; \mathcal{H}^\alpha)}. \end{aligned} \quad (3.11)$$

*Proof.* By symmetry, it suffices to show that

$$\begin{aligned} \|e^{it(\Delta_x - \Delta_{x'})}\gamma_0\|_{L_t^q(\mathbb{R}; L_x^r L_{x'}^2)} &\lesssim \|\gamma_0\|_{L_x^2 L_{x'}^2}, \\ \left\| \int_0^t e^{i(t-s)(\Delta_x - \Delta_{x'})} R(s) ds \right\|_{L_t^q(\mathbb{R}; L_x^r L_{x'}^2)} &\lesssim \|R(t)\|_{L_t^1(\mathbb{R}; L_x^2 L_{x'}^2)}. \end{aligned} \quad (3.12)$$

We denote by  $L^p(L^2)$  the collection of single variable  $L^p$ -functions  $f(x, \cdot) : \mathbb{R}^d \rightarrow L^2$ , which is identified with the collection of two spatial variable  $L_x^p L_{x'}^2$ -functions. It is obvious that by unitarity,

$$\|e^{it(\Delta_x - \Delta_{x'})}\gamma_0\|_{L^2(L^2)} = \|e^{it(\Delta_x - \Delta_{x'})}\gamma_0\|_{L_x^2 L_{x'}^2} = \|\gamma_0\|_{L_x^2 L_{x'}^2} = \|\gamma_0\|_{L^2(L^2)}. \quad (3.13)$$

On the other hand, by unitarity of the linear propagator  $e^{-it\Delta_{x'}}$ , the dispersive estimate  $\|e^{it\Delta_x}\|_{L_x^1 \rightarrow L_x^\infty} \lesssim |t|^{-d/2}$  and the Minkowski inequality, we get

$$\begin{aligned} \|e^{it(\Delta_x - \Delta_{x'})}\gamma_0\|_{L^\infty(L^2)} &= \|e^{it(\Delta_x - \Delta_{x'})}\gamma_0\|_{L_x^\infty L_{x'}^2} = \|e^{it\Delta_x}\gamma_0\|_{L_x^\infty L_{x'}^2} \\ &\leq \|e^{it\Delta_x}\gamma_0\|_{L_{x'}^2 L_x^\infty} \lesssim |t|^{-d/2} \|\gamma_0\|_{L_{x'}^2 L_x^1} \\ &\leq |t|^{-d/2} \|\gamma_0\|_{L_x^1 L_{x'}^2} = |t|^{-d/2} \|\gamma_0\|_{L^1(L^2)}. \end{aligned} \quad (3.14)$$

Then, (3.12) follows from the abstract version of Strichartz estimates (Theorem 10.1 in Keel and Tao [27]) with  $B_0 = L^2(L^2)$ ,  $B_1^* = L^\infty(L^2)$ ,  $H = L^2(L^2)$  and  $\sigma = \frac{d}{2}$ .  $\square$

*Remark 3.2.* The proof of Theorem 3.1 relies on the fact that an operator  $\gamma$  in  $\mathfrak{S}^2$  can be identified with its kernel  $\gamma(x, x')$  as a function in  $L_x^2 L_{x'}^2$ . An interesting open question is to derive similar Strichartz estimates for operators in different Schatten classes.

**3.2. Strichartz estimates for density functions.** Next, we establish the Strichartz estimates for density functions  $\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}$ .

**Theorem 3.3** (Strichartz estimates for density functions). *Suppose that*

$$\begin{cases} \alpha \geq 0 & \text{when } d = 1, \\ \alpha > \frac{d-1}{4} & \text{when } d \geq 2, \end{cases} \quad (3.15)$$

and

$$\begin{cases} \alpha_1 = \alpha & \text{when } d = 1, \\ \alpha_1 = 2\alpha - \frac{d-1}{2} & \text{when } d \geq 2 \text{ and } \frac{d-1}{4} < \alpha < \frac{d-1}{2}, \\ \alpha_1 < \frac{d-1}{2} & \text{when } d \geq 2 \text{ and } \alpha = \frac{d-1}{2}, \\ \alpha_1 = \alpha & \text{when } d \geq 2 \text{ and } \alpha > \frac{d-1}{2}. \end{cases} \quad (3.16)$$

(i) (*Homogeneous Strichartz estimate*)

$$\| |\nabla|^{\frac{1}{2}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} \|_{L_t^2(\mathbb{R}; H^{\alpha_1})} \lesssim \|\gamma_0\|_{\mathcal{H}^\alpha}. \quad (3.17)$$

(ii) (*Inhomogeneous Strichartz estimate*)

$$\| |\nabla|^{\frac{1}{2}} \rho \left[ \int_0^t e^{i(t-s)\Delta} R(s) e^{-i(t-s)\Delta} ds \right] \|_{L_t^2(\mathbb{R}; H^{\alpha_1})} \lesssim \|R(t)\|_{L_t^1(\mathbb{R}; \mathcal{H}^\alpha)}. \quad (3.18)$$

*Proof. (i):* We prove (i) by duality. The advantage of considering a dual inequality is that one can prove optimality with a small modification (see Proposition 3.4 and its proof).

We write the space-time Fourier transform of  $\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}$ ,

$$\begin{aligned}
(\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}})^\sim(\tau, \xi) &= \mathcal{F}_{t,x} \left\{ \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} e^{-it(|\xi_1|^2 - |\xi_2|^2)} \hat{\gamma}_0(\xi_1, \xi_2) e^{ix \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right\} \\
&= \mathcal{F}_t \left\{ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{-it(|\xi_1|^2 - |\xi_2|^2)} \hat{\gamma}_0(\xi_1, \xi_2) \delta(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2 \right\} \\
&= \mathcal{F}_t \left\{ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-it(|\xi_1|^2 - |\xi - \xi_1|^2)} \hat{\gamma}_0(\xi_1, \xi - \xi_1) d\xi_1 \right\} \\
&= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} \delta(\tau + |\xi_1|^2 - |\xi - \xi_1|^2) \hat{\gamma}_0(\xi_1, \xi - \xi_1) d\xi_1.
\end{aligned} \tag{3.19}$$

Then, by the Plancherel theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} (\langle \nabla \rangle^{\alpha_1} |\nabla|^{\frac{1}{2}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}})(x) \overline{V(t, x)} dx dt \\
&= \int_{\mathbb{R}^{d+1}} (\langle \nabla \rangle^{\alpha_1} |\nabla|^{\frac{1}{2}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}})^\sim(\tau, \xi) \overline{\tilde{V}(\tau, \xi)} d\xi d\tau \\
&= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d+1}} \langle \xi \rangle^{\alpha_1} |\xi|^{\frac{1}{2}} \delta(\tau + |\xi_1|^2 - |\xi - \xi_1|^2) \hat{\gamma}_0(\xi_1, \xi - \xi_1) \overline{\tilde{V}(\tau, \xi)} d\xi_1 d\xi d\tau \\
&= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} \langle \xi \rangle^{\alpha_1} |\xi|^{\frac{1}{2}} \hat{\gamma}_0(\xi_1, \xi - \xi_1) \overline{\tilde{V}(-|\xi_1|^2 + |\xi - \xi_1|^2, \xi)} d\xi_1 d\xi,
\end{aligned} \tag{3.20}$$

where  $\tau$  is integrated out in the last identity. Therefore, by duality, (3.17) is equivalent to the inequality

$$\left\| \frac{\langle \xi \rangle^{\alpha_1} |\xi|^{\frac{1}{2}} \tilde{V}(-|\xi_1|^2 + |\xi - \xi_1|^2, \xi)}{\langle \xi_1 \rangle^\alpha \langle \xi - \xi_1 \rangle^\alpha} \right\|_{L_{\xi, \xi_1}^2} \lesssim \|V\|_{L_{t \in \mathbb{R}}^2 L_x^2}. \tag{3.21}$$

We consider the square of the left hand side of (3.21),

$$I = \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} \frac{\langle \xi \rangle^{2\alpha_1} |\xi| |\tilde{V}(-2\xi \cdot \xi_1 + |\xi|^2, \xi)|^2}{\langle \xi_1 \rangle^{2\alpha} \langle \xi - \xi_1 \rangle^{2\alpha}} d\xi_1 \right\} d\xi. \tag{3.22}$$

In the one-dimensional case, we assume that  $\alpha_1 = \alpha$ . Then,  $\frac{\langle \xi \rangle^{2\alpha}}{\langle \xi_1 \rangle^{2\alpha} \langle \xi - \xi_1 \rangle^{2\alpha}}$  is bounded, since either  $|\xi_1|$  or  $|\xi - \xi_1|$  is greater than or equal to  $\frac{|\xi|}{2}$ . Thus, changing variables  $\tau = -2\xi\xi_1 + \xi^2$ , we get

$$\begin{aligned}
I &\lesssim \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |\xi| |\tilde{V}(-2\xi\xi_1 + \xi^2, \xi)|^2 d\xi_1 \right\} d\xi \\
&= \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{V}(\tau, \xi)|^2 d\tau d\xi = \frac{1}{2} \|V\|_{L_{t \in \mathbb{R}}^2 L_x^2}^2.
\end{aligned} \tag{3.23}$$

Suppose that  $d \geq 2$ . Given  $\xi \in \mathbb{R}^d$ , we introduce new variables  $q = q(\xi) = (q_1, \dots, q_d) \in \mathbb{R}^d$  such that  $\xi_1 = q_1 e_1 + \dots + q_d e_d$ , where  $\{e_j\}_{j=1}^d$  is an orthonormal basis in  $\mathbb{R}^d$  with  $e_1 = \frac{\xi}{|\xi|}$ . Then, changing variables by  $\xi_1 \mapsto q$ , we write

$$I = \int_{\mathbb{R}^{2d-1}} \left\{ \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2\alpha_1} |\xi| |\tilde{V}(-2|\xi|q_1 + |\xi|^2, \xi)|^2}{\langle q \rangle^{2\alpha} \langle q_1 - |\xi|, q' \rangle^{2\alpha}} dq_1 \right\} dq' d\xi, \tag{3.24}$$

where  $q' = (q_2, \dots, q_d)$  is a vector in  $\mathbb{R}^{d-1}$ . Next, we change variables  $\tau = -2|\xi|q_1 + |\xi|^2$ , and denote  $q_1^* = q_1(\tau, \xi) = -\frac{\tau - |\xi|^2}{2|\xi|}$ . Then,

$$\begin{aligned} I &= \int_{\mathbb{R}^{2d-1}} \left\{ \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2\alpha_1} |\tilde{V}(\tau, \xi)|^2}{2\langle (q_1^*, q') \rangle^{2\alpha} \langle (q_1^* - |\xi|, q') \rangle^{2\alpha}} d\tau \right\} dq' d\xi \\ &= \int_{\mathbb{R}^{d+1}} \left\{ \int_{\mathbb{R}^{d-1}} \frac{\langle \xi \rangle^{2\alpha_1} dq'}{2\langle (q_1^*, q') \rangle^{2\alpha} \langle (q_1^* - |\xi|, q') \rangle^{2\alpha}} \right\} |\tilde{V}(\tau, \xi)|^2 d\xi d\tau. \end{aligned} \quad (3.25)$$

Thus, in order to prove (3.21), it suffices to show that the integral  $\{\dots\}$  on the second line of (3.25) is bounded uniformly in  $\tau$  and  $\xi$ .

Suppose that  $|q_1^*| \leq \frac{|\xi|}{2}$  ( $\Rightarrow |q_1^* - |\xi|| \geq \frac{|\xi|}{2}$ ). Then,

$$\begin{aligned} &\int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1} dq'}{2\langle (q_1^*, q') \rangle^{2\alpha} \langle (q_1^* - |\xi|, q') \rangle^{2\alpha}} \\ &\leq \int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1} dq'}{2\langle q' \rangle^{2\alpha} \langle q_1^* - |\xi| \rangle^{2\alpha}} \\ &\leq \int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1}}{\langle q' \rangle^{2\alpha} \langle \xi \rangle^{2\alpha}} dq' = \int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1 - 2\alpha}}{\langle q' \rangle^{2\alpha}} dq' \\ &\lesssim \begin{cases} \langle \xi \rangle^{2\alpha_1 - 4\alpha + (d-1)} & \text{when } \alpha < \frac{d-1}{2}, \\ \langle \xi \rangle^{2\alpha_1 - (d-1)} \langle \ln |\xi| \rangle & \text{when } \alpha = \frac{d-1}{2}, \\ \langle \xi \rangle^{2\alpha_1 - 2\alpha} & \text{when } \alpha > \frac{d-1}{2}. \end{cases} \end{aligned} \quad (3.26)$$

On the other hand, if  $|q_1^*| \geq \frac{|\xi|}{2}$ , then

$$\begin{aligned} &\int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1} dq'}{2\langle (q_1^*, q') \rangle^{2\alpha} \langle (q_1^* - |\xi|, q') \rangle^{2\alpha}} \\ &\leq \int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1} dq'}{2\langle q_1^* \rangle^{2\alpha} \langle q' \rangle^{2\alpha}} \\ &\leq \int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1}}{\langle \xi \rangle^{2\alpha} \langle q' \rangle^{2\alpha}} dq' = \int_{|q'| \leq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1 - 2\alpha}}{\langle q' \rangle^{2\alpha}} dq' \\ &\lesssim \begin{cases} \langle \xi \rangle^{2\alpha_1 - 4\alpha + (d-1)} & \text{when } \alpha < \frac{d-1}{2}, \\ \langle \xi \rangle^{2\alpha_1 - (d-1)} \langle \ln |\xi| \rangle & \text{when } \alpha = \frac{d-1}{2}, \\ \langle \xi \rangle^{2\alpha_1 - 2\alpha} & \text{when } \alpha > \frac{d-1}{2}. \end{cases} \end{aligned} \quad (3.27)$$

For (3.26), (3.27) to be uniformly bounded in  $\tau$  and  $\xi$ , the exponent  $\alpha_1$  in (3.16) must be chosen so that

$$\begin{cases} 2\alpha_1 - 4\alpha + (d-1) = 0 & \text{when } \alpha < \frac{d-1}{2}, \\ 2\alpha_1 - (d-1) < 0 & \text{when } \alpha = \frac{d-1}{2}, \\ 2\alpha_1 - 2\alpha = 0 & \text{when } \alpha > \frac{d-1}{2}. \end{cases} \quad (3.28)$$

It remains to estimate the integral  $\{\dots\}$  in (3.25) whose integral domain is restricted to  $\{q' \in \mathbb{R}^{d-1} : |q'| \geq 2|\xi|\}$ . Now, using that both  $|(q_1^*, q')|$  and  $|(q_1^* - |\xi|, q')|$  are greater than

equal to  $|q'|$ , we prove that

$$\int_{|q'| \geq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1} dq'}{2\langle (q_1^*, q') \rangle^{2\alpha} \langle (q_1^* - |\xi|, q') \rangle^{2\alpha}} \leq \int_{|q'| \geq 2|\xi|} \frac{\langle \xi \rangle^{2\alpha_1}}{2\langle q' \rangle^{4\alpha}} dq' \lesssim \langle \xi \rangle^{2\alpha_1 - 4\alpha + (d-1)}, \quad (3.29)$$

where in the second inequality, we used the assumption  $\alpha > \frac{d-1}{4}$ . Furthermore, for all of the cases in (3.28), we have that the exponent on the r.h.s. in (3.29) satisfies

$$2\alpha_1 - 4\alpha + (d-1) \leq 0 \quad (3.30)$$

if concurrently,  $\alpha > \frac{d-1}{4}$  holds. Therefore, we conclude that the above integrals, (3.26), (3.27) and (3.29), are bounded uniformly in  $\tau$  and  $\xi$ .

This establishes (3.21) which is equivalent to (3.17).

(ii): By the Minkowski inequality and (i), we prove that

$$\begin{aligned} & \left\| |\nabla|^{\frac{1}{2}} \rho \left[ \int_0^t e^{i(t-s)\Delta} R(s) e^{-i(t-s)\Delta} ds \right] \right\|_{L_t^2(\mathbb{R}; H^{\alpha_1})} \\ & \leq \int_{\mathbb{R}} \left\| |\nabla|^{\frac{1}{2}} \rho_{e^{it\Delta}(e^{-is\Delta} R(s) e^{is\Delta}) e^{-it\Delta}} \right\|_{L_t^2(\mathbb{R}; H^{\alpha_1})} ds \\ & \lesssim \int_{\mathbb{R}} \|e^{-is\Delta} R(s) e^{is\Delta}\|_{\mathcal{H}^\alpha} ds = \int_{\mathbb{R}} \|R(s)\|_{\mathcal{H}^\alpha} ds = \|R(t)\|_{L_t^1(\mathbb{R}; \mathcal{H}^\alpha)}. \end{aligned} \quad (3.31)$$

This completes the proof.  $\square$

Now, we may prove optimality of the homogeneous Strichartz estimate (Theorem 3.3 (i)). Precisely, we will show that (3.17) fails in the case  $d \geq 2$  and  $\alpha = \frac{d-1}{4}$ , or the case  $d \geq 2$  and  $\alpha = \alpha_1 = \frac{d-1}{4}$ . Even more than that, we will show that in the first case, it is possible that  $\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}$  is not even a distribution (see (3.32)). We denote by  $(C_0^\infty)'$  the space of distributions, defined on  $\mathbb{R}^{d+1} = \mathbb{R}_t \times \mathbb{R}_x^d$ , that is dual to the space  $C_0^\infty = C_0^\infty(\mathbb{R}^{d+1})$  of compactly supported smooth functions.

**Proposition 3.4** (Optimality of the inequality (3.17)). *Let  $d \geq 2$ . Then,*

$$\sup_{\|\gamma_0\|_{\mathcal{H}^{\frac{d-1}{4}}} = 1} \frac{\| |\nabla_x|^{\frac{1}{2}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} \|_{(C_0^\infty)'}}{\|\gamma_0\|_{\mathcal{H}^{\frac{d-1}{4}}}} = \infty \quad (3.32)$$

and

$$\sup_{\|\gamma_0\|_{\mathcal{H}^{\frac{d-1}{2}}} = 1} \frac{\| |\nabla_x|^{\frac{1}{2}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} \|_{L_t^2(\mathbb{R}; \mathcal{H}^{\frac{d-1}{2}})}}{\|\gamma_0\|_{\mathcal{H}^{\frac{d-1}{2}}}} = \infty. \quad (3.33)$$

*Proof.* (3.32): By dualization as in Theorem 3.3, it is enough to show that the integral  $I$  (see (3.22)) is unbounded for  $V \in C_0^\infty(\mathbb{R}^{d+1})$ . Repeating (3.25), we write

$$I = \dots = \int_{\mathbb{R}^{d+1}} \left\{ \int_{\mathbb{R}^{d-1}} \frac{dq'}{2\langle (q_1^*, q') \rangle^{\frac{d-1}{2}} \langle (q_1^* - |\xi|, q') \rangle^{\frac{d-1}{2}}} \right\} |\tilde{V}(\tau, \xi)|^2 d\xi d\tau. \quad (3.34)$$

However, since the function

$$\frac{1}{\langle (q_1^*, q') \rangle^{\frac{d-1}{2}} \langle (q_1^* - |\xi|, q') \rangle^{\frac{d-1}{2}}} \quad (3.35)$$

is not integrable over  $\mathbb{R}^{d-1}$ , we conclude that  $I = \infty$ .

(3.33): By duality again, it suffices to find a bounded sequence  $\{V_n(t, x)\}_{n=1}^\infty$  in  $L_t^2(\mathbb{R}; L^2)$  such that

$$I_n = \left\| \frac{|\xi|^{\frac{1}{2}} \langle \xi \rangle^{\frac{d-1}{2}} \tilde{V}_n(-|\xi_1|^2 + |\xi - \xi_1|^2, \xi)}{\langle \xi_1 \rangle^{(d-1)/2} \langle \xi - \xi_1 \rangle^{(d-1)/2}} \right\|_{L_{\xi, \xi_1}^2}^2 \rightarrow \infty. \quad (3.36)$$

We define  $V_n(t, x)$  by

$$\tilde{V}_n(\tau, \xi) = \chi_1(\tau - |\xi|^2) \chi_2(\xi - n e_1), \quad (3.37)$$

where  $\chi_1(\tau) = \mathbf{1}_{[-1, 1]}(\tau)$  and  $\chi_2(\xi) = \mathbf{1}_{|\xi| \leq 1}$  are characteristic functions, and  $e_1 = (1, 0, \dots, 0)$  be a unit vector in  $\mathbb{R}^d$ . Then,

$$I_n = \dots = \int_{\mathbb{R}^{d+1}} \left\{ \int_{\mathbb{R}^{d-1}} \frac{\langle \xi \rangle^{d-1} dq'}{2 \langle (q_1^*, q') \rangle^{d-1} \langle (q_1^* - |\xi|, q') \rangle^{d-1}} \right\} |\tilde{V}_n(\tau, \xi)|^2 d\xi d\tau, \quad (3.38)$$

where  $q_1^* = -\frac{\tau - |\xi|^2}{2|\xi|}$ . Let  $n$  be sufficiently large. In the above integral,  $n - 1 \leq |\xi| \leq n + 1$ ,  $|q_1^*| = |\frac{\tau - |\xi|^2}{2|\xi|}| \leq \frac{1}{2(n-1)} \leq \frac{1}{2}$  and  $n - \frac{3}{2} \leq |q_1^* - |\xi|| = |\frac{\tau - |\xi|^2}{2|\xi|} + |\xi|| \leq n + \frac{3}{2}$ . Thus, we have

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \frac{\langle \xi \rangle^{d-1} dq'}{2 \langle (q_1^*, q') \rangle^{d-1} \langle (q_1^* - |\xi|, q') \rangle^{d-1}} &\sim \int_{\mathbb{R}^{d-1}} \frac{n^{d-1} dq'}{\langle q' \rangle^{d-1} (n^2 + |q'|^2)^{(d-1)/2}} \\ &\gtrsim \int_{|q'| \leq \frac{n}{2}} \frac{n^{d-1} dq'}{\langle q' \rangle^{d-1} n^{d-1}} \sim \ln n. \end{aligned} \quad (3.39)$$

Therefore, we conclude that

$$I_n \gtrsim \int_{\mathbb{R}^{d+1}} (\ln n) |\tilde{V}_n(\tau, \xi)|^2 d\xi d\tau \sim \ln n \rightarrow \infty, \quad (3.40)$$

since  $\|V_n\|_{L_t^2(\mathbb{R}; L^2)} \sim 1$ .  $\square$

#### 4. CONSERVATION OF RELATIVE ENERGY FOR REGULAR SOLUTIONS

In this section, we establish local well-posedness of the equation (1.18) in  $\mathfrak{H}^2$  (Theorem 4.2), where  $\mathfrak{H}^2$  is the Banach space of trace-class self-adjoint operators equipped with the norm

$$\|\gamma\|_{\mathfrak{H}^2} := \|\langle \nabla \rangle^2 \gamma \langle \nabla \rangle^2\|_{\mathfrak{S}^1}. \quad (4.1)$$

Subsequently, we show that  $\mathfrak{H}^2$ -regular solutions obey the conservation law for the relative energy (Proposition 4.3). The purpose is (as we recall from the outline of proof in Section 2) that we aim to approximate the solutions obtained in Theorem 2.4 with  $\mathfrak{H}^2$ -regular solutions, to prove the conservation of the relative energy of the former. This will be carried out in Section 6.

Before we give the statement of our main result regarding local well-posedness in  $\mathfrak{H}^2$ , we prove the following Lemma.

**Lemma 4.1.** (i) For  $Q \in \mathfrak{H}^2$ ,

$$\|[\rho_Q, \Pi^-]\|_{\mathfrak{S}^2} \lesssim \|Q\|_{\mathfrak{S}^2}. \quad (4.2)$$

(ii) For  $Q_1, Q_2 \in \mathfrak{H}^2$ ,

$$\|[\rho_{Q_1}, Q_2]\|_{\mathfrak{H}^2} \lesssim \|\rho_{Q_1}\|_{H^2} \|Q_2\|_{\mathfrak{H}^2} \lesssim \|Q_1\|_{\mathfrak{H}^2} \|Q_2\|_{\mathfrak{H}^2}. \quad (4.3)$$

*Proof.* (i) By the trivial inequality  $|[A, B]| \leq 2|AB|$  and the Leibnitz rule, we write

$$\begin{aligned} \|[\rho_Q, \Pi^-]\|_{\mathfrak{H}^2} &\leq 2\|(1 - \Delta)\rho_Q\Pi^-(1 - \Delta)\|_{\mathfrak{S}^1} \\ &\leq 2\left\{\|(-\Delta\rho_Q)(\Pi^-(1 - \Delta))\|_{\mathfrak{S}^1} + 2\sum_{k=1}^d\|(\partial_{x_k}\rho_Q)(\partial_{x_k}\Pi^-(1 - \Delta))\|_{\mathfrak{S}^1} \right. \\ &\quad \left. + \|\rho_Q((1 - \Delta)\Pi^-(1 - \Delta))\|_{\mathfrak{S}^1}\right\}. \end{aligned} \quad (4.4)$$

Then, using the Birman-Solomjak inequality (see Theorem 4.5 in [39])

$$\|f(x)g(-i\nabla)\|_{\mathfrak{S}^1} \lesssim \|f\|_{\ell^1 L^2} \|g\|_{\ell^1 L^2}, \quad (4.5)$$

where  $\|f\|_{\ell^1 L^2} = \sum_{z \in \mathbb{Z}^d} \|f\|_{L^2(C_z)}$  and  $\{C_z\}_{z \in \mathbb{Z}^d}$  is the collection of cubes  $C_z := z + [0, 1)^d$ , and using the eigenfunction expansion

$$\langle \nabla \rangle^2 Q \langle \nabla \rangle^2 = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle \langle \phi_j|,$$

where  $\{\phi_j\}_{j=1}^{\infty}$  is an orthonormal set in  $L^2$ , we obtain

$$\begin{aligned} \|[\rho_Q, \Pi^-]\|_{\mathfrak{H}^2} &\lesssim \|\Delta\rho_Q\|_{\ell^1 L^2} + \|\nabla\rho_Q\|_{\ell^1 L^2} + \|\rho_Q\|_{\ell^1 L^2} \\ &\lesssim \|(1 - \Delta)\rho_Q\|_{\ell^1 L^2} \\ &= \left\| (1 - \Delta) \sum_{j=1}^{\infty} \lambda_j |\langle \nabla \rangle^{-2} \phi_j|^2 \right\|_{\ell^1 L^2} \\ &\leq \sum_{j=1}^{\infty} |\lambda_j| \|(1 - \Delta)(|\langle \nabla \rangle^{-2} \phi_j|^2)\|_{\ell^1 L^2}. \end{aligned} \quad (4.6)$$

By the Leibnitz rule, the Hölder inequality and Sobolev inequality,

$$\begin{aligned} &\|(1 - \Delta)(|\langle \nabla \rangle^{-2} \phi_j|^2)\|_{\ell^1 L^2} \\ &\leq \|\langle \nabla \rangle^{-2} \phi_j\|_{\ell^2 L^4}^2 + 2\|\Delta\langle \nabla \rangle^{-2} \phi_j\|_{L^2} \|\langle \nabla \rangle^{-2} \phi_j\|_{\ell^2 L^\infty} + 2\|\nabla\langle \nabla \rangle^{-2} \phi_j\|_{\ell^2 L^4}^2 \\ &\lesssim \|\phi_j\|_{L^2}^2 = 1. \end{aligned} \quad (4.7)$$

Thus, we conclude that  $\|[\rho_Q, \Pi^-]\|_{\mathfrak{H}^2} \lesssim \sum_{j=1}^{\infty} |\lambda_j| = \|Q\|_{\mathfrak{H}^2}$ .



(ii) Similarly, by the Leibnitz rule, we write

$$\begin{aligned}
\|[\rho_{Q_1} Q_2]\|_{\mathfrak{H}^2} &\leq \|(\Delta \rho_{Q_1}) Q_2 \langle \nabla \rangle^2\|_{\mathfrak{S}^1} + 2\|(\nabla \rho_{Q_1}) \cdot \nabla Q_2 \langle \nabla \rangle^2\|_{\mathfrak{S}^1} + \|\rho_{Q_1} \langle \nabla \rangle^2 Q_2 \langle \nabla \rangle^2\|_{\mathfrak{S}^1} \\
&\leq \left\{ \|(\Delta \rho_{Q_1}) \langle \nabla \rangle^{-2}\|_{\text{Op}} + 2\|(\nabla \rho_{Q_1}) \langle \nabla \rangle^{-1}\|_{\text{Op}} + \|\rho_{Q_1}\|_{\text{Op}} \right\} \|Q_2\|_{\mathfrak{H}^2} \\
&\leq \left\{ \|\Delta \rho_{Q_1}\|_{L^2} \|\langle \nabla \rangle^{-2}\|_{L^2 \rightarrow L^\infty} + 2\|\nabla \rho_{Q_1}\|_{L^{d+}} \|\langle \nabla \rangle^{-1}\|_{L^{(\frac{2d}{d-2})^-} \rightarrow L^2} \right. \\
&\quad \left. + \|\rho_{Q_1}\|_{L^\infty} \right\} \|Q_2\|_{\mathfrak{H}^2} \\
&\lesssim \|(1 - \Delta) \rho_{Q_1}\|_{L^2} \|Q_2\|_{\mathfrak{H}^2} \quad (\text{by the Sobolev inequality}).
\end{aligned} \tag{4.8}$$

Then, using the canonical form as in (i), we complete the proof.  $\square$

The well-posedness result of this section can be formulated as follows:

**Theorem 4.2** (Local well-posedness in  $\mathfrak{H}^2$ ). *Let  $d = 2, 3$ . For initial data  $Q_0 \in \mathfrak{H}^2$ , there exists a unique solution  $Q \in C_t(I; \mathfrak{H}^2)$  to the equation (1.18).*

*Proof of Theorem 4.2.* We define

$$\Phi(Q) := e^{it\Delta} Q_0 e^{-it\Delta} - i \int_0^t e^{i(t-s)\Delta} [\rho_Q, \Pi^- + Q](s) e^{-i(t-s)\Delta} ds. \tag{4.9}$$

By the standard contraction mapping argument, it suffices to show that  $\Phi$  is contractive on the ball  $B_R \subset C_t(I; \mathfrak{H}^2)$  of radius  $R = 2\|Q_0\|_{\mathfrak{H}^2}$ , where  $I \ni 0$  is a short time interval to be chosen later.

First, by the Minkowski inequality and using the unitarity of the linear propagator, we write

$$\begin{aligned}
\|\Phi(Q)\|_{C_t(I; \mathfrak{H}^2)} &\leq \|Q_0\|_{\mathfrak{H}^2} + \|[\rho_Q, \Pi^-]\|_{L_t^1(I; \mathfrak{H}^2)} + \|[\rho_Q, Q]\|_{L_t^1(I; \mathfrak{H}^2)}, \\
\|\Phi(Q_1) - \Phi(Q_2)\|_{C_t(I; \mathfrak{H}^2)} &\leq \|[\rho_{(Q_1-Q_2)}, \Pi^-]\|_{L_t^1(I; \mathfrak{H}^2)} + \|[\rho_{(Q_1-Q_2)}, Q_1]\|_{L_t^1(I; \mathfrak{H}^2)} \\
&\quad + \|[\rho_{Q_2}, (Q_1 - Q_2)]\|_{L_t^1(I; \mathfrak{H}^2)},
\end{aligned} \tag{4.10}$$

where the identity

$$[\rho_{Q_1}, Q_1] - [\rho_{Q_2}, Q_2] = [\rho_{Q_1-Q_2}, Q_1] + [\rho_{Q_2}, Q_1 - Q_2]$$

is used for the difference. Then, applying Lemma 4.1, we get

$$\begin{aligned}
\|\Phi(Q)\|_{C_t(I; \mathfrak{H}^2)} &\leq \|Q_0\|_{\mathfrak{H}^2} + c|I| \|Q\|_{C_t(I; \mathfrak{H}^2)} + c|I| \|Q\|_{C_t(I; \mathfrak{H}^2)}^2, \\
\|\Phi(Q_1) - \Phi(Q_2)\|_{C_t(I; \mathfrak{H}^2)} &\leq c|I| \left\{ 1 + \|Q_1\|_{C_t(I; \mathfrak{H}^2)} + \|Q_2\|_{C_t(I; \mathfrak{H}^2)} \right\} \|Q_1 - Q_2\|_{C_t(I; \mathfrak{H}^2)}.
\end{aligned} \tag{4.11}$$

Now, choosing an interval  $I$  with length  $I = \min\{\frac{1}{4c}, \frac{1}{8c\|Q_0\|_{\mathfrak{H}^2}}\}$ , we conclude that

$$\begin{aligned}
\|\Phi(Q)\|_{C_t(I; \mathfrak{H}^2)} &\leq 2\|Q_0\|_{\mathfrak{H}^2}, \\
\|\Phi(Q_1) - \Phi(Q_2)\|_{C_t(I; \mathfrak{H}^2)} &\leq \frac{3}{4} \|Q_1 - Q_2\|_{C_t(I; \mathfrak{H}^2)}
\end{aligned} \tag{4.12}$$

for  $Q, Q_1, Q_2 \in B_R$ .  $\square$

Next, we prove the conservation law for the regular solutions.

**Proposition 4.3** (Conservation of the relative energy for regular solutions). *Let  $d = 2, 3$ . If  $Q(t) \in C_t(I; \mathfrak{H}^2)$  solves the equation (1.18) with initial data  $Q_0$ , then*

$$\mathcal{E}(Q(t)) = \mathcal{E}(Q_0), \quad \forall t \in I. \quad (4.13)$$

*Proof.* Differentiating the relative energy, and then inserting the equation (1.18), we get

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(Q(t)) &= \text{Tr} \left\{ (-\Delta - 1) \partial_t Q + \rho_Q \partial_t Q \right\} (t) \\ &= (-i) \text{Tr} \left\{ (-\Delta - 1) [-\Delta + \rho_Q, Q] + \rho_Q [-\Delta + \rho_Q, Q] \right\} (t). \end{aligned} \quad (4.14)$$

Indeed, the above calculation make sense, since  $Q(t) \in \mathfrak{H}^2$ . By cyclicity of the trace, we have  $\text{Tr}(-\Delta - 1)[- \Delta, Q] = 0$ ,  $\text{Tr}[\rho_Q, Q] = 0$  and  $\text{Tr} \rho_Q [\rho_Q, Q] = 0$ . Thus, we get

$$\frac{d}{dt} \mathcal{E}(Q(t)) = (-i) \left\{ \text{Tr}(-\Delta) [\rho_Q, Q] + \text{Tr} \rho_Q [-\Delta, Q] \right\} (t). \quad (4.15)$$

Then, by cyclicity again,

$$\begin{aligned} &\text{Tr}(-\Delta) [\rho_Q, Q] + \text{Tr} \rho_Q [-\Delta, Q] \\ &= \text{Tr}(-\Delta) \rho_Q Q - \text{Tr}(-\Delta) Q \rho_Q + \text{Tr} \rho_Q (-\Delta) Q - \text{Tr} \rho_Q Q (-\Delta) \\ &= \text{Tr}(-\Delta) \rho_Q Q - \text{Tr}(-\Delta) Q \rho_Q + \text{Tr}(-\Delta) Q \rho_Q - \text{Tr}(-\Delta) \rho_Q Q = 0. \end{aligned} \quad (4.16)$$

Therefore, we conclude that  $\frac{d}{dt} \mathcal{E}(Q(t)) = 0$ .  $\square$

## 5. LOCAL WELL-POSEDNESS IN A SPACE LARGER THAN THE ENERGY SPACE

In this section, we prove the main local well-posedness results contained in Theorem 2.4.

For the convenience of the reader, we review the norms introduced in Section 2. Let  $\alpha \geq 1$ . The initial data space  $\mathfrak{X}^\alpha$  was defined as the Banach space of self-adjoint operators equipped the norm

$$\|Q\|_{\mathfrak{X}^\alpha} := \|Q\|_{\text{Op}} + \|\Pi_2^+ Q\|_{\mathcal{H}^\alpha}. \quad (5.1)$$

Given  $I \subset \mathbb{R}$ , the solution space  $\mathfrak{Y}^\alpha(I)$  was defined as the Banach space of time-dependent self-adjoint operators with the norm

$$\|Q\|_{\mathfrak{Y}^\alpha(I)} := \|Q\|_{C_t(I; \text{Op})} + \|\Pi_2^+ Q\|_{\mathcal{S}^\alpha(I)} + \|\rho_Q\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)}, \quad (5.2)$$

where

$$\begin{cases} \eta > 0 & \text{when } d = 3 \text{ and } \alpha = 1, \\ \eta = 0 & \text{when } d = 2 \text{ and } \alpha \geq 1, \text{ or } d = 3 \text{ and } \alpha > 1. \end{cases} \quad (5.3)$$

The reason we need  $\eta$  in the norm is that  $\alpha_1$  in Theorem 3.3 is strictly less than 1 when  $\alpha = 1$  and  $d = 3$ . This is to offset the logarithmic growth of (3.26) and (3.27) in  $|\xi|$  which appears in this case.

Note that when  $\alpha = 1$ , Theorem 2.4 provides local well-posedness in a space larger than the relative kinetic energy space. Indeed, the initial data space  $\mathfrak{X}^1$  contains the relative kinetic energy space  $\mathcal{K}$ , since by the estimates in (A.7),

$$\|\Pi_2^+ Q\|_{\mathcal{H}^1} \leq \|\Pi_2^+ Q \Pi_2^+\|_{\mathfrak{H}^1} + \|\Pi_2^+ Q \Pi_2^-\|_{\mathcal{H}^1} < \infty. \quad (5.4)$$

The main tools to prove Theorem 2.4 are the Strichartz estimates in Section 3. Note that the Strichartz estimates in Theorem 3.3 cannot control the  $L^2$ -norm of (the low frequency part of) the density function  $\rho_{e^{it\Delta}Q_0e^{-it\Delta}}$ , while we need it for global well-posedness, since the potential energy of the relative energy is  $\|\rho_Q\|_{L^2}^2$ . Hence, we modify Strichartz estimates using the Lieb-Thirring inequality in Frank, Lewin, Lieb and Seiringer [21],

$$\mathrm{Tr}_0(-\Delta - 1)Q \geq K_{\mathrm{LT}} \int_{\mathbb{R}^d} \left\{ (\rho_{\Pi^-} + \rho_Q)^{1+\frac{2}{d}} - (\rho_{\Pi^-})^{1+\frac{2}{d}} - \frac{2+d}{d} (\rho_{\Pi^-})^{\frac{2}{d}} \rho_Q \right\} dx \quad (5.5)$$

and the generalized Sobolev inequality in [14]

$$\|\rho_Q\|_{L^2} \lesssim \|Q\|_{\mathcal{H}^1}. \quad (5.6)$$

**Proposition 5.1** (Local-in-time Strichartz estimates for density functions). *Let  $d = 2, 3$ ,  $\alpha \geq 1$  and  $\eta$  be as in (5.3), and let  $I \ni 0$  be an interval with  $|I| \leq 1$ . Then, we have*

$$\|\rho_{e^{it\Delta}Q_0e^{-it\Delta}}\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)} \lesssim \|Q_0\|_{\mathfrak{H}^\alpha} + \left\{ \mathrm{Tr}_0(-\Delta - 1)Q_0 \right\}^{\frac{1}{2}} \quad (5.7)$$

and

$$\left\| \rho \left[ \int_0^t e^{i(t-s)\Delta} R(s) e^{-i(t-s)\Delta} ds \right] \right\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)} \lesssim \|R(t)\|_{L_t^1(I; \mathcal{H}^\alpha)}. \quad (5.8)$$

*Proof.* We first prove (5.7). We shall estimate the l.h.s. of (5.7) via splitting the operator  $Q_0$  into: high-low  $\Pi_2^+ Q_0 \Pi_2^-$ , low-high  $\Pi_2^- Q_0 \Pi_2^+$ , high-high  $\Pi_2^+ Q_0 \Pi_2^+$  and low-low  $\Pi_2^- Q_0 \Pi_2^-$  frequency parts.

In order to consider high-low, low-high and high-high frequency parts of the operator  $Q_0$  we first observe that:

$$\Pi_2^+ Q_0 \Pi_2^- + (\Pi_2^- Q_0 \Pi_2^+ + \Pi_2^+ Q_0 \Pi_2^+) = \Pi_2^+ Q_0 \Pi_2^- + Q_0 \Pi_2^+. \quad (5.9)$$

Now we present the estimate that will be useful when we bound the terms coming from the r.h.s. of (5.9). More precisely, we have:

$$\begin{aligned} & \|\rho_{e^{it\Delta}Q_0e^{-it\Delta}}\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)} \\ & \lesssim \| |\nabla|^{\alpha+\frac{1}{2}-\eta} \rho_{e^{it\Delta}Q_0e^{-it\Delta}} \|_{L_t^2(I; L^2)} + \|\rho_{e^{it\Delta}Q_0e^{-it\Delta}}\|_{L_t^\infty(I; L^2)} \\ & \lesssim \|Q_0\|_{\mathcal{H}^\alpha}, \end{aligned} \quad (5.10)$$

where in the last inequality, we used the Strichartz estimate (3.17) for the first term, and Sobolev inequality (5.6) for the second term.

Then we estimate terms coming from the r.h.s. of (5.9) by using (5.10) as follows:

$$\begin{aligned} & \|\rho_{e^{it\Delta}(\Pi_2^+ Q_0 \Pi_2^- + Q_0 \Pi_2^+)e^{-it\Delta}}\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)} \\ & \lesssim \|\Pi_2^+ Q_0 \Pi_2^- + Q_0 \Pi_2^+\|_{\mathcal{H}^\alpha} \\ & \leq \|\Pi_2^+ Q_0 \Pi_2^-\|_{\mathcal{H}^\alpha} + \|Q_0 \Pi_2^+\|_{\mathcal{H}^\alpha} \\ & \leq 2\|\Pi_2^+ Q_0\|_{\mathcal{H}^\alpha} \quad (\text{by symmetry}). \end{aligned} \quad (5.11)$$

It remains to consider the low-low frequency part  $\Pi_2^- Q_0 \Pi_2^-$ . We claim that in general, the Sobolev norm of the density function of the low-low frequency part is bounded by its  $L^2$  norm. In other words,

$$\|\rho_{\Pi_2^- Q_0 \Pi_2^-}\|_{H^\alpha} \lesssim \|\rho_{\Pi_2^- Q_0 \Pi_2^-}\|_{L^2}. \quad (5.12)$$

Indeed, by the Plancherel theorem,

$$\begin{aligned} \|\rho_{\Pi_2^- Q_0 \Pi_2^-}\|_{H^\alpha} &= \left\| \langle \xi \rangle^\alpha \int_{\mathbb{R}^d} (\Pi_2^- Q_0 \Pi_2^-)^\wedge(\xi - \xi', \xi') d\xi' \right\|_{L_\xi^2} \\ &\lesssim \left\| \int_{\mathbb{R}^d} (\Pi_2^- Q_0 \Pi_2^-)^\wedge(\xi - \xi', \xi') d\xi' \right\|_{L_\xi^2} = \|\rho_{\Pi_2^- Q_0 \Pi_2^-}\|_{L^2}, \end{aligned} \quad (5.13)$$

where in the first inequality, we used  $|\xi - \xi'|, |\xi'| \leq 2 \Rightarrow |\xi| \leq 4$ . Therefore, using (5.12) (with the fact that  $e^{\pm it\Delta}$  commutes with  $\Pi_2^-$ ), the estimate (A.4) (which follows from the Lieb-Thirring inequality) and unitarity of the linear propagator, we prove

$$\begin{aligned} &\|\rho_{e^{it\Delta} \Pi_2^- Q_0 \Pi_2^- e^{-it\Delta}}\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)} \\ &\lesssim \|\rho_{e^{it\Delta} \Pi_2^- Q_0 \Pi_2^- e^{-it\Delta}}\|_{L_t^\infty(I; L^2)} \\ &\lesssim \sup_{t \in I} \left\{ \text{Tr}_0(-\Delta - 1) e^{it\Delta} \Pi_2^- Q_0 \Pi_2^- e^{-it\Delta} \right\}^{\frac{1}{2}} \\ &= \left\{ \text{Tr}_0(-\Delta - 1) \Pi_2^- Q_0 \Pi_2^- \right\}^{\frac{1}{2}} \leq \left\{ \text{Tr}_0(-\Delta - 1) Q_0 \right\}^{\frac{1}{2}}. \end{aligned} \quad (5.14)$$

This establishes the proof of (5.7).

Now we prove (5.8). By the Minkowski inequality and (5.10), we prove that the left hand side of (5.8) is bounded by

$$\begin{aligned} &\int_I \|\rho_{e^{it\Delta} (e^{-is\Delta} R(s) e^{is\Delta}) e^{-it\Delta}}\|_{L_t^2(I; H^{\alpha+\frac{1}{2}-\eta}) \cap L_t^\infty(I; L^2)} ds \\ &\lesssim \int_I \|e^{-is\Delta} R(s) e^{is\Delta}\|_{\mathcal{H}^\alpha} ds = \|R(t)\|_{L_t^1(I; \mathcal{H}^\alpha)}. \end{aligned} \quad (5.15)$$

□

The following lemma is analogous to Lemma 4.1.

**Lemma 5.2.** *Let  $\alpha \geq 1$ . For  $Q, Q_1, Q_2 \in \mathfrak{Y}^\alpha(I)$ , we have the following:*

(i)

$$\|[\rho_{Q_1}, Q_2]\|_{L_t^1(I; \mathcal{H}^\alpha)} \lesssim |I|^{1/8} \|\rho_{Q_1}\|_{L_t^2(I; H^\alpha)} \|Q_2\|_{S^\alpha(I)}. \quad (5.16)$$

(ii)

$$\|[\rho_Q, \Pi^-]\|_{L_t^1(I; \mathcal{H}^\alpha)} \lesssim |I|^{\frac{1}{2}} \|\rho_Q\|_{L_t^2(I; H^\alpha)}. \quad (5.17)$$

*Proof.* (i) First, we consider the term with the low-low frequency part of  $Q_2$ , that is,  $\Pi_2^- Q_2 \Pi_2^-$ . To do that we recall the definition of the Hilbert-Schmidt type Sobolev space

$\mathcal{H}^\alpha$  (see (2.8)) and use Hölder inequality as follows:

$$\begin{aligned}
& \|[\rho_{Q_1}, \Pi_2^- Q_2 \Pi_2^-]\|_{L_t^1(I; \mathcal{H}^\alpha)} \\
& \leq 2\|\rho_{Q_1} \Pi_2^- Q_2 \Pi_2^-\|_{L_t^1(I; \mathcal{H}^\alpha)} \\
& = 2\|\langle \nabla \rangle^\alpha \rho_{Q_1} \Pi_2^- Q_2 \Pi_2^- \langle \nabla \rangle^\alpha\|_{L_t^1(I; \mathfrak{S}^2)} \\
& \leq 2\|\langle \nabla \rangle^\alpha \rho_{Q_1} \Pi_2^-\|_{L_t^1(I; \mathfrak{S}^2)} \|Q_2\|_{C_t(I; \text{Op})} \|\Pi_2^- \langle \nabla \rangle^\alpha\|_{\text{Op}} \\
& = 2\|\langle \nabla_x \rangle^\alpha (\rho_{Q_1}(x) \Pi_2^-(x - x'))\|_{L_t^1(I; L_x^2 L_{x'}^2)} \|Q_2\|_{C_t(I; \text{Op})} \|\Pi_2^- \langle \nabla \rangle^\alpha\|_{\text{Op}}.
\end{aligned} \tag{5.18}$$

Hence, by the fractional Leibnitz rule (Theorem A.8 in [28]), we get

$$\begin{aligned}
\|[\rho_{Q_1}, \Pi_2^- Q_2 \Pi_2^-]\|_{L_t^1(I; \mathcal{H}^\alpha)} & \lesssim |I|^{\frac{1}{2}} \|\rho_{Q_1}\|_{L_t^2(I; H^\alpha)} \|\Pi_2^-(x)\|_{H^\alpha} \|Q_2\|_{C_t(I; \text{Op})} \\
& \lesssim |I|^{\frac{1}{2}} \|\rho_{Q_1}\|_{L_t^2(I; H^\alpha)} \|Q_2\|_{C_t(I; \text{Op})},
\end{aligned} \tag{5.19}$$

where  $\Pi_2^-(x)$  is the kernel of the operator  $\Pi_2^-$ .

Next, we consider the remainder with  $\tilde{Q}_2 := Q_2 - \Pi_2^- Q_2 \Pi_2^-$ . By the fractional Leibnitz rule (Theorem A.8 in [28]) and the Sobolev inequality,

$$\begin{aligned}
\|[\rho_{Q_1}, \tilde{Q}_2]\|_{\mathcal{H}^\alpha} & \leq 2\|\rho_{Q_1} \tilde{Q}_2\|_{\mathcal{H}^\alpha} \\
& = 2\|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha (\rho_{Q_1}(x) \tilde{Q}_2(x, x'))\|_{L_x^2 L_{x'}^2} \\
& \lesssim \|\rho_{Q_1}\|_{H^\alpha} \|\langle \nabla_{x'} \rangle^\alpha \tilde{Q}_2\|_{L_x^\infty L_{x'}^2} + \|\rho_{Q_1}\|_{L_x^{\frac{2d}{d-2+\epsilon}}} \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \tilde{Q}_2\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} \\
& \lesssim \|\rho_{Q_1}\|_{H^\alpha} \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \tilde{Q}_2\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2},
\end{aligned} \tag{5.20}$$

where  $\epsilon > 0$  is sufficiently small. Then, further splitting  $\tilde{Q}_2$

$$\tilde{Q}_2 = (\Pi_2^+ Q_2 \Pi_2^+ + \Pi_2^+ Q_2 \Pi_2^-) + \Pi_2^- Q_2 \Pi_2^+ = \Pi_2^+ Q_2 + \Pi_2^- Q_2 \Pi_2^+$$

and dropping  $\Pi_2^-$  in the second term,

$$\begin{aligned}
& \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \tilde{Q}_2\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} \\
& \leq \|\rho_{Q_1}\|_{H^\alpha} \left\{ \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \Pi_2^+ Q_2\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} + \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \Pi_2^- Q_2 \Pi_2^+\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} \right\} \\
& \leq \|\rho_{Q_1}\|_{H^\alpha} \left\{ \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \Pi_2^+ Q_2\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} + \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha Q_2 \Pi_2^+\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} \right\} \\
& = 2\|\rho_{Q_1}\|_{H^\alpha} \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \Pi_2^+ Q_2\|_{L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2} \quad (\text{by symmetry}).
\end{aligned} \tag{5.21}$$

Combining (5.20) and (5.21), we conclude after performing Hölder with respect to  $t$  that

$$\begin{aligned}
\|[\rho_{Q_1}, Q_2]\|_{L_t^1(I; \mathcal{H}^\alpha)} & \lesssim |I|^{\frac{4-d-\epsilon}{4}} \|\rho_{Q_1}\|_{L_t^2(I; H^\alpha)} \|\langle \nabla_x \rangle^\alpha \langle \nabla_{x'} \rangle^\alpha \Pi_2^+ Q_2\|_{L_t^{\frac{4}{d-2+\epsilon}}(I; L_x^{\frac{2d}{2-\epsilon}} L_{x'}^2)} \\
& \leq |I|^{1/8} \|\rho_{Q_1}\|_{L_t^2(I; H^\alpha)} \|\Pi_2^+ Q_2\|_{S^\alpha(I)},
\end{aligned} \tag{5.22}$$

where the last inequality follows thanks to  $(\frac{4}{d-2+\epsilon}, \frac{2d}{2-\epsilon})$  being an admissible pair. Hence (5.16) is proved.

(ii) Similarly, we prove that

$$\begin{aligned}
\|[\rho_Q, \Pi^-]\|_{L_t^1(I; \mathcal{H}^\alpha)} &\leq 2\|\rho_Q \Pi^-\|_{L_t^1(I; \mathcal{H}^\alpha)} \\
&= 2\|\langle \nabla_x \rangle^\alpha (\rho_Q(x) (\Pi^- \langle \nabla \rangle^\alpha)(x - x'))\|_{L_t^1(I; L_x^2 L_{x'}^2)} \\
&\lesssim |I|^{\frac{1}{2}} \|\rho_Q\|_{L_t^2(I; H^\alpha)} \|\Pi^-(x)\|_{H^{2\alpha}} \\
&\lesssim |I|^{\frac{1}{2}} \|\rho_Q\|_{L_t^2(I; H^\alpha)}.
\end{aligned} \tag{5.23}$$

□

*Proof of Theorem 2.4.* Let  $\alpha \geq 1$ . Let  $I \ni 0$  be a sufficiently short interval ( $|I| \leq 1$ ) whose length will be chosen later. We define the nonlinear mapping  $\Phi$  by

$$\Phi(Q) = e^{it\Delta} Q_0 e^{-it\Delta} - i \int_0^t e^{i(t-s)\Delta} [\rho_Q, \Pi^- + Q](s) e^{-i(t-s)\Delta} ds. \tag{5.24}$$

Then, by a standard contraction mapping argument, it suffices to show that  $\Phi$  is contractive on a ball in  $\mathfrak{Y}^\alpha(I)$ .

Applying trivial estimates for  $\Phi(Q)$  in the operator norm and Strichartz estimates (Theorem 3.1 for  $\Pi_2^+ \Phi(Q)$  and Proposition 5.1 for  $\rho_{\Phi(Q)}$ ), we get

$$\begin{aligned}
\|\Phi(Q)\|_{\mathfrak{Y}^\alpha(I)} &= \|\Phi(Q)\|_{C_t(I; \text{Op})} + \|\Pi_2^+ \Phi(Q)\|_{S^\alpha(I)} + \|\rho_{\Phi(Q)}\|_{L_t^\infty(I; L^2) \cap L_t^2(I; H_x^{\alpha+\frac{1}{2}-\eta})} \\
&\lesssim \left\{ \|Q_0\|_{\text{Op}} + \|[\rho_Q, \Pi^- + Q]\|_{L_t^1(I; \text{Op})} \right\} \\
&\quad + \left\{ \|\Pi_2^+ Q_0\|_{\mathcal{H}^\alpha} + \|\Pi_2^+ [\rho_Q, \Pi^- + Q]\|_{L_t^1(I; \mathcal{H}^\alpha)} \right\} \\
&\quad + \left\{ A + \|[\rho_Q, \Pi^- + Q]\|_{L_t^1(I; \mathcal{H}^\alpha)} \right\} \\
&\lesssim A + \|[\rho_Q, \Pi^-]\|_{L_t^1(I; \mathcal{H}^\alpha)} + \|[\rho_Q, Q]\|_{L_t^1(I; \mathcal{H}^\alpha)},
\end{aligned} \tag{5.25}$$

where

$$A = \|Q_0\|_{\mathfrak{X}^\alpha} + \{\text{Tr}_0(-\Delta - 1)Q_0\}^{\frac{1}{2}}. \tag{5.26}$$

Then, by Lemma 5.2, we obtain

$$\|\Phi(Q)\|_{\mathfrak{Y}^\alpha(I)} \leq cA + c|I|^{\frac{1}{2}} \|\rho_Q\|_{L_t^2(I; H^\alpha)} + c|I|^{1/8} \|\rho_Q\|_{L_t^2(I; H^\alpha)} \|Q\|_{S^\alpha(I)}. \tag{5.27}$$

In the same way, we show that

$$\begin{aligned}
&\|\Phi(Q_1) - \Phi(Q_2)\|_{\mathfrak{Y}^\alpha(I)} \\
&= \left\| \int_0^t e^{i(t-s)\Delta} \left\{ [\rho_{Q_1-Q_2}, \Pi^- + Q_1](s) + [\rho_{Q_2}, Q_1 - Q_2](s) \right\} e^{-i(t-s)\Delta} ds \right\|_{S^\alpha(I)} \\
&\leq c|I|^{\frac{1}{2}} \|\rho_{(Q_1-Q_2)}\|_{L_t^2(I; H^\alpha)} + c|I|^{1/8} \|\rho_{(Q_1-Q_2)}\|_{L_t^2(I; H^\alpha)} \|Q_1\|_{S^\alpha(I)} \\
&\quad + c|I|^{1/8} \|\rho_{Q_2}\|_{L_t^2(I; H^\alpha)} \|Q_1 - Q_2\|_{S^\alpha(I)}.
\end{aligned} \tag{5.28}$$

Now, let  $B_R$  be the ball of radius  $R$  in  $\mathfrak{Y}^\alpha(I)$ , where  $R = 2cA$  and  $|I| = \min\{\frac{1}{(4c)^2}, \frac{1}{(8c^2A)^8}\}$ . The norms in the bounds of (5.27) and (5.28) are bounded by their  $\mathfrak{Y}^\alpha$ -norms (see (5.2)).

Therefore, it follows that

$$\begin{aligned} \|\Phi(Q)\|_{\mathfrak{H}^\alpha(I)} &\leq R, \\ \|\Phi(Q_1) - \Phi(Q_2)\|_{\mathfrak{H}^\alpha(I)} &\leq \frac{3}{4}\|Q_1 - Q_2\|_{\mathfrak{H}^\alpha(I)} \end{aligned} \quad (5.29)$$

if  $Q, Q_1, Q_2 \in B_R$ . Therefore, we conclude that  $\Phi$  is a contraction on  $B_R$ .  $\square$

## 6. CONSERVATION OF THE RELATIVE ENERGY IN THE KINETIC ENERGY SPACE

In this section we establish the conservation of the relative energy for solutions evolved initially from the relative kinetic energy space, by proving Theorem 2.5.

We prove Theorem 2.5 by approximating  $Q(t) \in \mathfrak{H}^1(I)$  by a sequence of regular solutions using the following approximation lemma.

**Lemma 6.1** (Approximation lemma). *Let  $d = 2, 3$ . For  $Q_0 \in \mathcal{K}$ , there exists a sequence  $\{Q^{(n)}\}_{n=1}^\infty$  of finite-rank smooth operators such that  $-\Pi^- \leq Q^{(n)} \leq \Pi^+$ ,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \|Q^{(n)} - Q\|_{\text{Op}} &= 0, \\ \lim_{n \rightarrow \infty} \|\Delta + 1|^{\frac{1}{2}}(Q^{(n)} - Q)^{\pm\pm}|\Delta + 1|^{\frac{1}{2}}\|_{\mathfrak{S}^1} &= 0, \\ \lim_{n \rightarrow \infty} \|\rho_{Q^{(n)}} - \rho_Q\|_{L^2} &= 0, \end{aligned} \quad (6.1)$$

where  $Q^{\pm\pm} = \Pi_1^\pm Q \Pi_1^\pm$ .

*Proof.* It suffices to construct a sequence of Hilbert-Schmidt operators with the desired properties. Let

$$P_n = \mathbf{1}_{(\frac{1}{n} \leq |\Delta+1| \leq n)} \quad (6.2)$$

be the frequency cut-off away from the Fermi sphere  $\{\xi \in \mathbb{R}^d : |\xi| = 1\}$ , and define

$$Q^{(n)} := P_n Q P_n. \quad (6.3)$$

It is shown in [21, Lemma 3.2] and [33, Lemma 5] that

$$\begin{aligned} \|Q^{(n)} - Q\|_{\text{Op}} &\rightarrow 0 \\ \|\Delta + 1|^{\frac{1}{2}}(Q^{(n)} - Q)|\Delta + 1|^{\frac{1}{2}}\|_{\mathfrak{S}^1} &\rightarrow 0 \\ \rho_{\Pi_2^-(Q^{(n)} - Q)\Pi_2^-} &\rightarrow 0 \text{ in } L^2. \end{aligned} \quad (6.4)$$

As a consequence, by the Sobolev inequality (5.6),

$$\begin{aligned} \|\rho_{\Pi_2^+(Q^{(n)} - Q)\Pi_2^+}\|_{L^2} &\lesssim \|\Pi_2^+(Q^{(n)} - Q)\Pi_2^+\|_{\mathcal{H}^1} \\ &\lesssim \|\Delta + 1|^{\frac{1}{2}}(Q^{(n)} - Q)^{++}|\Delta + 1|^{\frac{1}{2}}\|_{\mathfrak{S}^1} \rightarrow 0. \end{aligned} \quad (6.5)$$

Note that

$$\Pi_2^+(Q - Q^{(n)})\Pi_2^- = \mathbf{1}_{(|\Delta+1| \geq n)} Q \Pi_2^- + \Pi_2^+ P_n Q \mathbf{1}_{(|\Delta+1| \leq \frac{1}{n})}. \quad (6.6)$$

Hence, by the Sobolev inequality (5.6) again,

$$\begin{aligned} \|\rho_{\Pi_2^+(Q^{(n)}-Q)\Pi_2^-}\|_{L^2} &\lesssim \|\mathbf{1}_{(|\Delta+1|\geq n)}Q\Pi_2^-\|_{\mathcal{H}^1} + \|\Pi_2^+P_nQ\mathbf{1}_{(|\Delta+1|\leq \frac{1}{n})}\|_{\mathcal{H}^1} \\ &\lesssim \|\Delta+1\|^{\frac{1}{2}}\mathbf{1}_{(|\Delta+1|\geq n)}Q\|_{\mathfrak{S}^2} + \|\Delta+1\|^{\frac{1}{2}}Q\mathbf{1}_{(|\Delta+1|\leq \frac{1}{n})}\|_{\mathfrak{S}^2} \rightarrow 0, \end{aligned} \quad (6.7)$$

since  $|\Delta+1|^{\frac{1}{2}}Q \in \mathfrak{S}^2$  if  $Q \in \mathcal{K}$ . Similarly, we show that  $\rho_{\Pi_2^-(Q^{(n)}-Q)\Pi_2^+} \rightarrow 0$  in  $L^2$ .  $\square$

In order to approximate a solution  $Q(t) \in \mathfrak{Y}^1(I)$  suitably by a sequence of regular solutions  $\{Q^{(n)}(t)\}_{n=1}^\infty$ , we need to justify that the interval  $I^{(n)}$  of existence for  $Q^{(n)}(t)$  in  $\mathfrak{H}^2$  does not shrink to  $\{0\}$  as  $n \rightarrow \infty$ . The following lemma asserts that the interval of existence in a regular space can be extended to the interval of existence in the relative kinetic energy space.

**Lemma 6.2** (Existence time). *Suppose that  $Q_0 \in \mathfrak{H}^2 \cap \mathcal{K}$ . Let  $Q(t)$  be the solution to the equation (1.18) with initial data  $Q_0$  satisfying  $\|Q(t)\|_{\mathfrak{Y}^1(I)} < \infty$ , where  $I$  is the interval of existence given by Theorem 2.4. Then,  $Q(t) \in C_t(I; \mathfrak{H}^2)$ .*

*Proof. Step 1.* We claim that it suffices to show that

$$\|\rho_Q\|_{L_t^2(I; H^2)} < \infty. \quad (6.8)$$

Indeed, by the local well-posedness theorem in the regular space  $\mathfrak{H}^2$  (Theorem 4.2), there exists a short interval  $I' \subset I$  such that  $Q(t)$  exists in  $C_t(I'; \mathfrak{H}^2)$ . Let  $t \in I'$ . Then, applying Lemma 4.1 (i) and the first inequality in Lemma 4.1 (ii) to the Duhamel formula

$$Q(t) = e^{it\Delta}Q_0e^{-it\Delta} - i \int_0^t e^{i(t-s)\Delta}[\rho_Q, \Pi^- + Q](s)e^{-i(t-s)\Delta}ds, \quad (6.9)$$

we get

$$\|Q(t)\|_{\mathfrak{H}^2} \leq \|Q_0\|_{\mathfrak{H}^2} + c \int_0^t \{1 + \|\rho_Q(s)\|_{H^2}\} \|Q(s)\|_{\mathfrak{H}^2} ds. \quad (6.10)$$

Hence, by the Gronwall inequality, we obtain

$$\begin{aligned} \|Q(t)\|_{\mathfrak{H}^2} &\leq \|Q_0\|_{\mathfrak{H}^2} \exp \left\{ c \int_0^t (1 + \|\rho_Q(s)\|_{H^2}) ds \right\} \\ &\leq \|Q_0\|_{\mathfrak{H}^2} \exp \left\{ c|I| + c|I|^{\frac{1}{2}} \|\rho_Q\|_{L_t^2(I; H^2)} \right\} \quad (\text{by } |I'| \leq |I|) \\ &< \infty \quad (\text{by (6.8)}). \end{aligned} \quad (6.11)$$

Thanks to this a priori bound, we can extend the interval  $I'$  of existence to  $I$ , since  $Q(t)$  does not blow up in  $\mathfrak{H}^2$  on the interval  $I$ . Therefore, we conclude that  $Q(t) \in C_t(I; \mathfrak{H}^2)$ .

*Step 2.* In order to prove (6.8), we choose intervals  $I$  and  $I'$  as follows. Let  $I$  be the interval of existence in the proof of Theorem 2.4 with  $\alpha = 1$ . Then

$$\begin{aligned} \|Q\|_{\mathfrak{Y}^1(I)} &\leq 2cA, \\ |I| &= \min \left\{ \frac{1}{(4c)^2}, \frac{1}{(8c^2A)^8} \right\}, \end{aligned}$$



with

$$A = \|Q_0\|_{\mathfrak{X}^1} + \{\text{Tr}(-\Delta - 1)Q_0\}^{\frac{1}{2}}.$$

Let  $I' \subset I$  be a shorter interval such that the solution  $Q(t)$  exists in  $\mathfrak{Y}^{\frac{3}{2}-\eta}(I')$  with small  $\eta > 0$ . We claim that there exists  $C_1 > 0$ , depending on  $I$ , and not on  $I'$ , such that

$$\|\rho_Q\|_{L_t^2(I'; H^{\frac{3}{2}-\eta})} \leq \|Q\|_{\mathfrak{Y}^{\frac{3}{2}-\eta}(I')} \leq C_1 < \infty. \quad (6.12)$$

Indeed, applying Strichartz estimates to the Duhamel formula which leads (5.27) with  $\alpha = \frac{3}{2} - \eta$ , we get

$$\|Q\|_{\mathfrak{Y}^{\frac{3}{2}-\eta}(I')} \leq cA' + c|I'|^{\frac{1}{2}}\|\rho_Q\|_{L_t^2(I'; H^{\frac{3}{2}-\eta})} + c|I'|^{\frac{1}{8}}\|\rho_Q\|_{L_t^2(I'; H^{\frac{3}{2}-\eta})}\|Q\|_{\mathfrak{S}^{\frac{3}{2}-\eta}(I')}, \quad (6.13)$$

where

$$A' = \|Q_0\|_{\mathfrak{X}^{\frac{3}{2}-\eta}} + \{\text{Tr}(-\Delta - 1)Q_0\}^{\frac{1}{2}}.$$

Hence, it follows from the choice of  $I'(\subset I)$  and the definition of the  $\mathfrak{Y}^\alpha$ -norm (see (5.2)) that

$$\begin{aligned} \|Q\|_{\mathfrak{Y}^{\frac{3}{2}-\eta}(I')} &\leq cA' + c|I|^{\frac{1}{2}}\|\rho_Q\|_{L_t^2(I; H^{\frac{3}{2}-\eta})} + c|I|^{\frac{1}{8}}\|\rho_Q\|_{L_t^2(I; H^{\frac{3}{2}-\eta})}\|Q\|_{\mathfrak{S}^{\frac{3}{2}-\eta}(I')} \\ &\leq cA' + c|I|^{\frac{1}{2}}\|Q\|_{\mathfrak{Y}^1(I)} + c|I|^{\frac{1}{8}}\|Q\|_{\mathfrak{Y}^1(I)}\|Q\|_{\mathfrak{Y}^{\frac{3}{2}-\eta}(I')} \\ &\leq cA' + \frac{cA}{2} + \frac{1}{4}\|Q\|_{\mathfrak{Y}^{\frac{3}{2}-\eta}(I')}. \end{aligned} \quad (6.14)$$

Hence,  $\|Q\|_{\mathfrak{Y}^{\frac{3}{2}-\eta}(I')} \leq 2cA' = C_1$ .

Next, we improve (6.12) to  $\|Q\|_{\mathfrak{Y}^{\frac{3}{2}+}(I')} < \infty$ . Let  $I'' \subset I'$  be a sufficiently short interval such that the solution  $Q(t)$  exists in  $\mathfrak{Y}^{\frac{3}{2}+}(I'')$ . For  $t \in I''$ , applying trivial estimates to the Duhamel formula for  $Q$ ,

$$\|\Pi_2^+ Q(t)\|_{\mathcal{H}^{\frac{3}{2}+}} \leq \|Q_0\|_{\mathcal{H}^{\frac{3}{2}+}} + 2 \int_0^t \|\rho_Q(s)\Pi^- \|_{\mathcal{H}^{\frac{3}{2}+}} + \|\rho_Q Q(s)\|_{\mathcal{H}^{\frac{3}{2}+}} ds. \quad (6.15)$$

Using fractional Leibnitz rule (Theorem A.8 in [28]) as in the proof of Lemma 5.2, one can show that

$$\|\rho_Q(s)\Pi^- \|_{\mathcal{H}^{\frac{3}{2}+}} = \|\langle \nabla_{x'} \rangle^{\frac{3}{2}+} (\rho_Q(x) \langle \nabla \rangle^{\frac{3}{2}+} \Pi^-)(x - x')\|_{L_x^2 L_{x'}^2} \lesssim \|\rho_Q\|_{\mathcal{H}^{\frac{3}{2}+}}. \quad (6.16)$$

Similarly,

$$\begin{aligned} \|\rho_Q Q\|_{\mathcal{H}^{\frac{3}{2}+}} &\leq \|\rho_Q \Pi_2^- Q \Pi_2^-\|_{\mathcal{H}^{\frac{3}{2}+}} + \|\rho_Q(Q - \Pi_2^- Q \Pi_2^-)\|_{\mathcal{H}^{\frac{3}{2}+}} \\ &= \|\langle \nabla \rangle^{\frac{3}{2}+} \rho_Q \Pi_2^- Q \Pi_2^- \langle \nabla \rangle^{\frac{3}{2}+}\|_{\mathfrak{S}^2} \\ &\quad + \|\langle \nabla_x \rangle^{\frac{3}{2}+} (\rho_Q(x) \langle \nabla_{x'} \rangle^{\frac{3}{2}+} (Q - \Pi_2^- Q \Pi_2^-)(x, x'))\|_{L_x^2 L_{x'}^2} \\ &\lesssim \|\langle \nabla \rangle^{\frac{3}{2}+} \rho_Q \Pi_2^- \|_{\mathfrak{S}^2} \|Q\|_{\text{Op}} \|\Pi_2^- \langle \nabla \rangle^{\frac{3}{2}+}\|_{\text{Op}} \\ &\quad + \|\rho_Q\|_{\mathcal{H}^{\frac{3}{2}+}} \|\langle \nabla_{x'} \rangle^{\frac{3}{2}+} (Q - \Pi_2^- Q \Pi_2^-)(x, x')\|_{L_x^\infty L_{x'}^2} \\ &\quad + \|\rho_Q\|_{L^\infty} \|\langle \nabla_x \rangle^{\frac{3}{2}+} \langle \nabla_{x'} \rangle^{\frac{3}{2}+} (Q - \Pi_2^- Q \Pi_2^-)(x, x')\|_{L_x^2 L_{x'}^2}. \end{aligned} \quad (6.17)$$

Then, by (6.16) for  $\langle \nabla \rangle^{\frac{3}{2}+} \rho_Q \Pi_2^-$  and the Sobolev inequality for  $Q - \Pi_2^- Q \Pi_2^- = \Pi_2^+ Q + \Pi_2^- Q \Pi_2^+$ ,

$$\begin{aligned} \|\rho_Q Q\|_{\mathcal{H}^{\frac{3}{2}+}} &\lesssim \|\rho_Q\|_{H^{\frac{3}{2}+}} + \|\rho_Q\|_{H^{\frac{3}{2}+}} \|Q - \Pi_2^- Q \Pi_2^-\|_{\mathcal{H}^{\frac{3}{2}+}} \\ &\leq \|\rho_Q\|_{H^{\frac{3}{2}+}} \left\{ 1 + \|\Pi_2^+ Q\|_{\mathcal{H}^{\frac{3}{2}+}} + \|\Pi_2^- Q \Pi_2^+\|_{\mathcal{H}^{\frac{3}{2}+}} \right\} \\ &\leq \|\rho_Q\|_{H^{\frac{3}{2}+}} \left\{ 1 + 2\|\Pi_2^+ Q\|_{\mathcal{H}^{\frac{3}{2}+}} \right\} \quad (\text{by symmetry}). \end{aligned} \quad (6.18)$$

Thus, it follows that

$$\|\Pi_2^+ Q(t)\|_{\mathcal{H}^{\frac{3}{2}+}} \leq \|Q_0\|_{\mathcal{H}^{\frac{3}{2}+}} + c \int_0^t \|\rho_Q(s)\|_{H^{\frac{3}{2}+}} \left\{ 1 + 2\|\Pi_2^+ Q(s)\|_{\mathcal{H}^{\frac{3}{2}+}} \right\} ds. \quad (6.19)$$

Therefore, by Gronwall's inequality, we conclude that

$$\begin{aligned} \|\Pi_2^+ Q(t)\|_{\mathcal{H}^{\frac{3}{2}+}} &\leq (1 + \|Q_0\|_{\mathcal{H}^{\frac{3}{2}+}}) \exp \left\{ \int_0^t c \|\rho_Q(s)\|_{H^{\frac{3}{2}+}} ds \right\} \\ &\leq (1 + \|Q_0\|_{\mathcal{H}^{\frac{3}{2}+}}) \exp \left\{ c |I|^{1/2} \|\rho_Q\|_{L_t^2(I; H^{\frac{3}{2}+})} \right\} \quad (\text{by } I'' \subset I) \\ &\leq (1 + \|Q_0\|_{\mathcal{H}^{\frac{3}{2}+}}) \exp \left\{ c |I|^{1/2} C_1 \right\} =: C_2 < \infty \quad (\text{by (6.12)}) \end{aligned} \quad (6.20)$$

Finally, applying Strichartz estimates (Theorem 3.1) to the Duhamel formula as in the proof of Theorem 2.4 and then using (6.16) and (6.18), we get

$$\begin{aligned} \|Q\|_{\mathfrak{Y}^{\frac{3}{2}+}(I'')} &\lesssim A'' + \int_0^t \|\rho_Q(s) \Pi_2^-\|_{\mathcal{H}^{\frac{3}{2}+}} + \|\rho_Q Q(s)\|_{\mathcal{H}^{\frac{3}{2}+}} ds \\ &\lesssim A'' + \int_0^t \|\rho_Q(s)\|_{H^{\frac{3}{2}+}} \left\{ 1 + 2\|\Pi_2^+ Q(s)\|_{\mathcal{H}^{\frac{3}{2}+}} \right\} ds \\ &\leq A'' + |I|^{1/2} \|\rho_Q\|_{L_t^2(I; H^{\frac{3}{2}+})} (1 + C_2) \quad (\text{by (6.20)}) \\ &\leq A'' + |I|^{1/2} C_1 (1 + C_2) =: C_3 \quad (\text{by (6.12)}), \end{aligned} \quad (6.21)$$

where

$$A'' = \|Q_0\|_{\mathfrak{X}^{\frac{3}{2}+}} + \{\text{Tr}(-\Delta - 1)Q_0\}^{\frac{1}{2}}.$$

Moreover, by the definition of the norm,  $\|\rho_Q\|_{L_t^2(I''; H^2)} \leq C_3$ . Note that this bound is independent of  $I'' \subset I$ , and that  $Q(t) \in \mathcal{K}$  for all  $t \in I''$  by the conservation of relative energy (see Step 1 with (6.21)). Thus, by Theorem 2.4, we can extend the interval  $I''$  to  $I$  with the desired bound  $\|\rho_Q\|_{L_t^2(I; H^2)} \leq C_3$ .  $\square$

Now we are ready to prove the relative energy conservation law.

*Proof of Theorem 2.5.* Let  $Q(t) \in \mathcal{Y}(I)$  be the solution to the equation (1.18) with initial data  $Q_0 \in \mathcal{K}$ , where  $I = [-T_1, T_2]$ . Let  $Q^{(n)}(t)$  be the solutions with initial data  $Q_0^{(n)}$ , where  $\{Q_0^{(n)}\}_{n=1}^\infty$  is a sequence of initial data approximating  $Q_0$ , obtained from Lemma 6.1. For arbitrary  $\epsilon > 0$ , we assume that  $n$  is sufficiently large so that  $Q^{(n)}(t) \in C(I^\epsilon; \mathfrak{H}^2)$  with

$I^\epsilon = [-T_1 + \epsilon, T_2 - \epsilon]$ . Then, it follows from continuity of the data-to-solution map from  $\mathfrak{X}^1$  to  $\mathfrak{Y}^1(I^\epsilon)$  that

$$\lim_{n \rightarrow \infty} \|Q^{(n)}(t) - Q(t)\|_{\mathfrak{Y}^1(I^\epsilon)} = 0. \quad (6.22)$$

In particular, we have

$$\lim_{n \rightarrow \infty} \|\rho_{Q^{(n)}(t)} - \rho_{Q(t)}\|_{L^\infty(I^\epsilon; L^2)} = 0. \quad (6.23)$$

Moreover, for any  $t \in I^\epsilon$ , by weak semi-continuity of  $\text{Tr}(-\Delta - 1)Q$ , we have

$$\text{Tr}(-\Delta - 1)Q(t) \leq \liminf_{n \rightarrow \infty} \text{Tr}(-\Delta - 1)Q^{(n)}(t). \quad (6.24)$$

Hence, it follows that

$$\begin{aligned} \mathcal{E}(Q(t)) &= \text{Tr}(-\Delta - 1)Q(t) + \frac{1}{2} \int_{\mathbb{R}^d} (\rho_{Q(t)})^2 dx \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \text{Tr}(-\Delta - 1)Q^{(n)}(t) + \frac{1}{2} \int_{\mathbb{R}^d} (\rho_{Q^{(n)}(t)})^2 dx \right\}. \end{aligned} \quad (6.25)$$

Then, using the conservation law for regular solutions and Lemma 6.1, we prove that

$$\begin{aligned} \mathcal{E}(Q(t)) &\leq \liminf_{n \rightarrow \infty} \left\{ \text{Tr}(-\Delta - 1)Q_0^{(n)} + \frac{1}{2} \int_{\mathbb{R}^d} (\rho_{Q_0^{(n)}})^2 dx \right\} \\ &= \left\{ \text{Tr}(-\Delta - 1)Q_0 + \frac{1}{2} \int_{\mathbb{R}^d} (\rho_{Q_0})^2 dx \right\} = \mathcal{E}(Q_0). \end{aligned} \quad (6.26)$$

Repeating the above backward in time, we prove that  $\mathcal{E}(Q_0) \leq \mathcal{E}(Q(t))$ . Thus, we conclude that  $\mathcal{E}(Q_0) = \mathcal{E}(Q(t))$ .  $\square$

## 7. PROOF OF THE MAIN THEOREM (THEOREM 2.3)

We present the proof for positive time only. By the conservation of the relative energy,  $Q(t) \in \mathcal{K}$  for all  $t \in [0, T^+)$ , where  $T^+$  is the maximal existence time (in the positive direction) for  $Q$  given in the Theorem 2.4. Suppose that  $T^+ < \infty$ . Then, we evolve  $Q(t)$  from the time  $T^+ - \epsilon$  with sufficiently small  $\epsilon > 0$ . However, since the size of the interval of existence has a lower bound depending only on

$$\text{Tr}_0(-\Delta - 1)Q(T^+ - \epsilon) \leq \mathcal{E}(Q(T^+ - \epsilon)) = \mathcal{E}(Q_0),$$

we can deduce a contradiction.

## APPENDIX A. RELATIVE ENERGY

**Lemma A.1.** *Let  $d = 2, 3$ . Suppose that  $Q \in \mathcal{K}$ .*

(i) *The relative kinetic energy  $\text{Tr}_0(-\Delta - 1)Q$  is positive. Moreover,*

$$\text{Tr}_0(-\Delta - 1)Q = \sum_{\pm} \| |\Delta + 1|^{\frac{1}{2}} Q^{\pm\pm} | \Delta + 1|^{\frac{1}{2}} \|_{\mathfrak{S}^1}. \quad (A.1)$$

(ii) *The density function  $\rho_Q$  satisfies*

$$\|\rho_Q\|_{L^2} \lesssim \text{Tr}_0(-\Delta - 1)Q + \left\{ \text{Tr}_0(-\Delta - 1)Q \right\}^{\frac{1}{2}}. \quad (A.2)$$

Therefore, the relative energy  $\mathcal{E}(Q)$  is finite.

*Proof.* (i): (A.1) follows from the definition of the relative kinetic energy (see (2.3)), since  $Q^{++} = \Pi^+(\gamma - \Pi^-)\Pi^+ = \Pi^+\gamma\Pi^+ \geq 0$  and  $Q^{--} = \Pi^-(\gamma - \Pi^-)\Pi^- = \Pi^-(\gamma - 1)\Pi^- \leq 0$ .

(ii): Applying the Lieb-Thirring inequality in Frank, Lewin, Lieb and Seiringer (5.5) to  $\Pi_2^- Q \Pi_2^- = \Pi_2^- \gamma \Pi_2^- - \Pi^-$  and using the Taylor series expansion for  $(1+x)^{1+\frac{2}{d}} - 1 - \frac{2+d}{d}x$ , we get

$$\mathrm{Tr}_0(-\Delta - 1)\Pi_2^- Q \Pi_2^- \gtrsim \int_{\mathbb{R}^d} \min \left\{ (\rho_{\Pi_2^- Q \Pi_2^-})^2, (\rho_{\Pi_2^- Q \Pi_2^-})^{1+\frac{2}{d}} \right\} dx. \quad (\text{A.3})$$

However, since  $-\Pi^- \leq \Pi_2^- Q \Pi_2^- = \Pi_2^- \gamma \Pi_2^- - \Pi^- \leq \Pi_2^-$ , we have  $|\rho_{\Pi_2^- Q \Pi_2^-}(x)| \lesssim 1$ . Hence, it follows from Lemma A.1 (i) that

$$\|\rho_{\Pi_2^- Q \Pi_2^-}\|_{L^2}^2 \lesssim \mathrm{Tr}_0(-\Delta - 1)Q. \quad (\text{A.4})$$

On the other hand, by the generalized Sobolev inequality (5.6), we get

$$\|\rho_{(Q - \Pi_2^- Q \Pi_2^-)}\|_{L^2} \lesssim \|Q - \Pi_2^- Q \Pi_2^-\|_{\mathcal{H}^1} \leq \|\Pi_2^+ Q \Pi_2^+\|_{\mathcal{H}^1} + 2\|\Pi_2^+ Q \Pi_2^-\|_{\mathcal{H}^1}. \quad (\text{A.5})$$

It is obvious that by Lemma A.1 (i),

$$\|\Pi_2^+ Q \Pi_2^+\|_{\mathcal{H}^1} \lesssim \| |\Delta + 1|^{\frac{1}{2}} Q^{++} |\Delta + 1|^{\frac{1}{2}} \|_{\mathfrak{S}^2} \leq \mathrm{Tr}_0(-\Delta - 1)Q. \quad (\text{A.6})$$

Moreover,

$$\|\Pi_2^+ Q \Pi_2^-\|_{\mathcal{H}^1}^2 \lesssim \| |\Delta + 1|^{\frac{1}{2}} Q \|^2_{\mathfrak{S}^2} = \mathrm{Tr} |\Delta + 1|^{\frac{1}{2}} Q^2 |\Delta + 1|^{\frac{1}{2}} \leq \mathrm{Tr}_0(-\Delta - 1)Q, \quad (\text{A.7})$$

because

$$\begin{aligned} Q^{++} - Q^{--} - Q^2 &= \Pi_1^+(\gamma - \Pi_1^-)\Pi_1^+ - \Pi_1^-(\gamma - \Pi_1^-)\Pi_1^- - (\gamma - \Pi_1^-)^2 \\ &= \dots = \gamma(1 - \gamma) \geq 0. \end{aligned} \quad (\text{A.8})$$

Collecting all, we complete the proof.  $\square$

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**Conflict of Interest.** The authors declare that they have no conflict of interest

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